DEPARTMENT OF ECONOMICS, FINANCE AND ADMINISTRATION

Analysis of Highway Networks: A Linear Programming Formulation

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Highway problems have been analyzed in a variety of ways and each has yielded important practical and theoretical results. Utilization of linear programming techniques offers another way to solve highway problems and such a statement is the purpose of the present paper. The problem used as a vehicle for the analysis is the minimization of joint costs of transportation and investment in a highway system. Transportation routes are defined as directed arcs linking urban centers. The restraint set recognizes requirements for transportation capacity to and from urban centers, the relationships between traffic on any route and capacity on other routes, as well as budgetary limits on investment. The dual problem is also examined.

A special attempt is made to elaborate economic, geographic, and engineering implications of the problem in addition to implications of a general pedagogical and theoretical sort. One result of this analysis follows from an examination of the efficiency prices of the dual. These prices clarify concepts of benefits to property owners and vehicle users. Also, they offer a guide to alternate investment possibilities.

Several ways to continue this type of analysis are suggested, such as the inclusion of intersections and congestion costs in the arrangement of routes. In addition, the formulation suggests priorities for empirical research.

• A FOREMOST FEATURE of any complex transportation system is the high level of spatial interdependence among its component routes. Many empirical examples attest to this fact, for instance, the diversion of traffic from one route to another following changes in the capacity of routes. A more subtle but more significant illustration of interdependence occurs when route improvements change the competitive positions of producing and consuming centers and, thus, traffic patterns. This influence of the improvement of routes in the transportation network on the strategic position of nodal points or areas of supply and demand cannot be over-emphasized. Here is a basic feature of the transport revolution. Urban centers, regions, and nations tend to grow or decline as their comparative advantage via the transportation system waxes and wanes. This basic interdependence presents one of the many problems which must be recognized in the analysis of transportation networks.

In any transportation system, analysis is the basis for development decisions and consequent investment choices between alternate modes of transportation and alternate routes, with opportunities available for investments varying in degree. In addition to the interdependence previously noted, analysis for development decisions is complicated because there is no product of transportation priced on the market and there is little place to start with the traditional analysis by estimation of supply and demand curves. The strength of the previous statement varies, of course, between different modes of transportation. With most of the highway network there is no valued product on the market at all; whereas certain other systems, where prices are associated with the transportation service, operate under a structure of prices which is extremely difficult to consider market-derived.

At the present time the Federal government has embarked upon a long-range, high-cost program of basic improvements in the interstate highway system. One feature of this program is the nearly complete absence of pricing and related market guide-posts for the use of decision makers in guiding this enormous development program. The present paper is motivated by this problem of improvements in the interstate highway system, while taking cognizance of the spatial interdependence of routes and the problem of pricing services of the highway system. The study uses linear programming, a tool used in a 1956 Highway Research Board study by LaVallee (*9*). Methods other than linear programming are available for the analysis of network problems. These were reviewed recently by Kalaba (*8*).

It is presumed in the problem discussed here that the choice has already been made to invest in the transportation system in general and in the highway system in particular. The specific problem, then, is to chose among alternate investments, taking into account the fact that investment will affect producing and consuming areas and thus will change the flow situation within the network. What criteria can be used to select among alternate investment choices? How can it be assured that the most efficient transportation possible is being purchased with the budget adopted? How large a budget should be adopted? How is a just and equitable taxation policy to be devised? These questions overlap in scope, but they generally identify the problem for discussion.

The need for guidance in previous similar situations has served as the motivation for the formulation and careful statement of the characteristics of several transportation problems. For example, Bevis (*2*) recently studied the cost-benefit aspects of expressways. However, many of these previous statements have concerned only specific parts of the whole and have not been general statements. One consequence of this theoretical fragmentation is the existence of a variety of theories, each of which has its advocates and opponents; a situation which stems in part from the right of the analyst to criticize, but which also arises from deficiencies in basic theoretical outlooks as to the nature of transportation problems.

Tinbergen (*14*) has pointed out that problems of transportation have been, in general, peripheral to the central considerations of economics. Thus, there is little in present economic theory that contributes to the solution of the present problem. In spite of this, Nicholson (*11*) and Beckman (*1*) recently formulated several highway questions in economic terms. Other workers, such as Flood (*5*), have pointed out how certain transportation problems of an elementary type resist simple analysis. Flood notes, for example, that the solution of the traveling salesman problem (that is, the selection of a shortest route between a series of nodal points) is empirically extremely laborious in all but the most trivial cases. It is not surprising that in Crane's (*3*) recent review of transportation research little was noted in reference to highway problems.

In light of the scanty theoretical resources available and the great empirical difficulties involved, the present paper may be considered somewhat ambitious. The goal of the analysis presented here is an exclusive scheme displaying the economic characteristics of transportation with special reference to highway networks. Many of the statements in the analysis are exploratory. They represent an attempt to set down the significant interrelations in a transportation network. Other statements are explanatory. The latter especially occur when minimization and maximization techniques are discussed. Still other statements are intended to be suggestive. These suggestive statements relate to needed research and value or benefit implications of the analysis.

CONTENT OF A GENERAL STATEMENT

The nature of the analysis of highway networks has been noted in a gross way. What should a more realistic statement encompass? If a general statement is indeed intended to cover all of the questions of interest of a "choice among alternates" type, it must incorporate some explicit measure which may be empirically enumerated by the researcher and used as a basis for actual decisions. In the present discussion dollar value is used as a measure of the value of alternate actions.

The second requirement of a general statement is that it include all of the pertinent characteristic features of the system which it is intended to represent. It is essential to recognize the network character of the highway system; in this case a network of routes connecting points between which traffic may move. A *route* is here defined as a link between points of origin and destination of traffic. A definition of nodal points follows from the definition of routes, *nodal points* being places of origin and destination of goods or persons. Later this latter definition is broadened to include the activity of transferring goods at route intersections. The analysis thus incorporates spatially separated centers of production and consumption, and spatial competition for resources and markets over transportation routes.

The two items just discussed seem essential to measuring the suitability of the alternates suggested by the analysis and to displaying the spatial character of the problem as well as the special production and consumption characteristics of the situation. Additional considerations can be introduced into the problem to bring the outline of the scheme into clearer operational focus. These are as follows:

1. The supply and demand parameters within which the economic system operates.
2. Pertinent costs of transportation and transportation facilities.
3. Points of production and routes of marketing for all commodities or services which use transportation.
4. Geographic realities of the orientation of routes and locations of nodal points.
5. Changes with time.

In the ensuing discussion these characteristics are introduced formally into the analysis in varying degrees.

FORMULATION OF THE MODEL

The discussion of characteristics which should be recognized in a realistic analytic model of highway transportation was to a large extent pragmatic. These items are elements of the author's impressions of reality, but these intuitive views are strengthened by statements of others and by considerable empirical experience. Pragmatic impressions are one resource available for model construction. The second resource at hand is the tool of linear programming. Here is an optimization technique which has been found suitable for many problems of the type similar to that presented here.

In the present section of the analysis the pragmatic elements of the model * are brought together within the formal analytic structure. The first portion discusses the way variables are recognized; the second portion discusses the formalization of the primal linear programming problem; the third portion discusses the formalization of the dual or pricing problem.

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* The model discussed here is one of several developed at the SSRC Workshop on Linear Economic Models, held at Stanford University during the summer of 1957.

Identification of Variables

Essential to the concept of a transportation network is movement along routes between nodal points. Let x_{ij} be the amount shipped from place i to place j. x may be measured in any units (for instance, tons per year or passengers per day). Let i represent the originating place with the number of such places ranging from 1 to n. j will represent the receiving place, of which there are also n (j ranging from 1 to n). A receiving place and originating place may be the same geographically. Thus, places found in the list of originating nodes may also appear in the list of receiving nodes. In the present model, traffic between such identical nodes is not recognized as moving along a transportation route (that is, $x_{ii}=0$).

Consider two urban centers and the routes connecting them. Call one urban center place 1 and the other urban center place 2. Center 1 could produce and ship to 2 and this shipment would be measured in units x_{12}. The same figure indicates, of course, the receipts of place 2 from place 1. For instance, x_{21} would indicate both the receipts of 1 from 2 and the outflow of production of place 2 which is directed to place 1. It is to be noted that there are two links between places 1 and 2 — a link for flows in one direction and another link for flows in the opposite direction. Is this assertion a departure from the real conditions of the highway network? It seemingly is in practice a departure, but it is not necessary that capacity on road routes be the same in both directions. Prevailing practice of making capacity in both directions approximately equal may, in fact, be in error.

Now let attention be shifted from a system that contains only two urban centers, or nodes on the transportation network, to a transportation network with many centers. Considered geographically, the shift is to a scatter of points on a plane each of which connects with all other points by a system of directed routes (Fig. 1). One way to utilize the definition just given is to number each urban center and then identify each directed route. In a system of n urban centers there would be $n(n-1)$ directed routes.

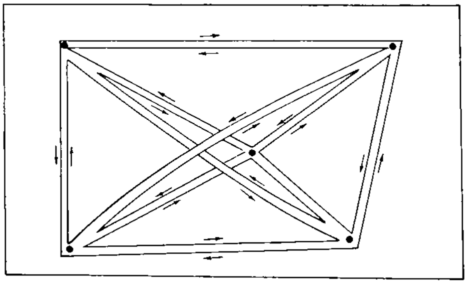

Figure 1. A completely connected network.

How does this system differ from the interstate highway system? In reality not every place is connected with every other place by a direct link. From the standpoint of the layout of the network, there are apparently two reasons for this. One is the configuration of the area that contains the network. For example, land routes from the northeastern United States to Minnesota are restrained from direct connections by the Great Lakes. The other feature is that, in general, places are linked directly only to adjoining places and not to more distant places. The original network (Fig. 1) is characterized by many intersections of direct routes which go relatively long distances. The key difference between this pattern and the actual reproduction of a real pattern (Fig. 2) is that each intersection on the real network is occupied by an urban center.

Commodities shipped between two places which are not directly linked are shipped via transfer points. For example,

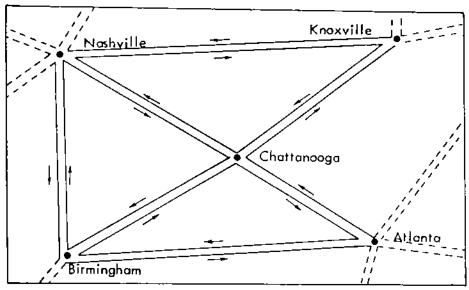

Figure 2. A segment of the interstate highway system.

commodities might be shipped from place 1 to place 4 when the directed route is via place 2 and place 3. In the present scheme the movement x_{14} would not be identified. The movement would appear as part of the movement x_{12}, part of the movement x_{23}, and part of the movement x_{34}. Movements are direct only between places which are close geographically.

This labelling of routes is a distinguishing feature of the present scheme. It tends to duplicate actual conditions in real transportation networks and it avoids the problem of congestion at intersections in these networks. Aspects of this problem will be discussed later. On the other hand, the effect of commodity flow on production and consumption in urban centers may be somewhat obscured by this classification system. This will also be elaborated later.

In addition to measures of movements, other variables may be attached to the routes from i to j. Each route has a capacity as well as an actual movement. The capacity on the route from i to j can be designated c_{ij}. This capacity would be measured in the same units as the units of x_{ij}.

In addition to capacity and amount of flow, an average cost of transportation may be assigned to the directed route from i to j. This can be designated r_{ij} and is taken to be independent of the flow, x_{ij}. This measure would be in cost per unit moved from i to j. Thus $r_{ij}x_{ij}$ is the cost of transportation at some particular time for the total movement between i and j. $\sum_i \sum_j r_{ij}x_{ij}$ is, then, the total cost of transportation in the network.

Urban centers contribute traffic to the system and receive traffic from the system. These amounts are related to their productive capacity and demand. Take the i^{th} place and call its total production A_i. The j^{th} place has some level of demand, B_j. A_i and B_j are here defined as levels of production and demand at the onset of the analysis.

Emphasis thus far has been on urban centers and routes. There has been no attempt to state what type of goods is being hauled along the route. The commodity superscript ϕ may be added to the notation to indicate the commodity hauled; for example, x_{ij}^{ϕ} (where $\phi = 1$ to m) is the amount of the ϕ^{th} commodity moved from i to j. However, to simplify the notation this convention will not be formally adopted here. It should be noted that the ϕ^{th} commodity is implicit and the discussion holds for that commodity.

To summarize, the notation adopted is as follows:

x_{ij} = Amount shipped from i to j; $x_{ii}=0$; i to j is a directed arc between urban centers; in general, $x_{ij} \neq x_{ji}$;
c_{ij} = Capacity of the i to j route;
r_{ij} = Cost of transportation from i to j;
A_i = Original production at the i^{th} source;
B_j = Original demand at the j^{th} place; and
ϕ^{th} Commodity implicit.

The motivating and underlying assumption of the whole analysis is that investment in the transportation network is warranted and desired. It is undecided what the level of investment will be; that is, how much will be put into the system. Determining this amount requires considerations beyond those of the transportation system itself. Questions pertinent here are those of returns from transportation *versus* returns from other great activities, say education. Although the determination of the level of transportation investment is beyond the present model, the present formulation contributes to the decision process by offering a measure of return as expected from various levels of input. One other feature of the system as yet undetermined is the allocation of the gross input into the road system among alternates within the system. Choices may be made to increase the capacity of any particular route or any particular mix of routes and to increase each to some unique level.

The problem of charging for capacity increases is also unsolved. This is basic to the choice among alternate investments and is also basic to problems of tariff establishment and collection. In some cases development of the highway network requires actual tolls, but more often one is concerned with problems of taxation policy.

The physical characteristics of the transportation system related to these latter problems must be identified. It should be recalled that the transportation system warrants investment and the decision has been made to make investments. Investment will cause capacity increases on routes. Any route may have its capacity increased and may be identified by an amount K_{ij}, which is the increment in capacity along the route from i to j. This would be measured in the same units used for measuring c_{ij}. Each increment in the capacity would have attached to it a cost P_{ij} representing the cost to increase the capacity by one unit along the route i to j. $P_{ij}K_{ij}$ is the cost of increased capacity along the route from i to j.

Formulation of the Problem

The transportation system is congested and it is desired to expand capacity. Only on a most general level might one attempt to expand capacity to a level which, considering alternate investments in the economy, will maximize national product. Some suggested ways for this maximization might be determined from available input-output information on the American economy; however, in a large measure the determination of the optimum level of investment in transportation is obscure. Dosages of transportation inputs might be divided among many places and among many transportation routes. Decisions of this sort are obscured when one deals with the transportation sector in input-output models aggregated for the whole economy. Isard (7) has noted the problems of using input-output data on a less than national scale.

When it is known that inputs will be made to transportation, the problem may be formulated at a lower level than that of the problem just noted: the level of alternates among transportation systems and alternate inputs among places within any given transportation system. This problem will be formulated here for the road system as a minimization problem, namely to minimize

$$\Delta = \sum_i \sum_j (P_{ij}K_{ij} + r_{ij}x_{ij}) \qquad (1)$$

The goal identified is to minimize the joint cost of new capacity and the cost of transportation. The value of $P_{ij}K_{ij}$ summed over all i and all j yields the cost of all capacity additions to the system. The value of $r_{ij}x_{ij}$ summed over all i and j gives the total cost of transportation in the system. The K_{ij}'s and the x_{ij}'s are the variables that are to be chosen to achieve the minimization (the *choice variables*). The objective function by itself has little meaning. It is now necessary to identify the restraints under which minimization values of K_{ij} and x_{ij} are chosen.

It may be suggested that the minimal value of this expression, termed the objective function, occurs when no capacity is added and only transportation costs are incurred. As the problem has been discussed, however, this could not be true because it has been assumed that there is motivation for investment in new capacity; that is, it is desired to bring all capacity in line with actual movement requirements. This requirement may be formalized in the system as one of the restraints to which the objective function is subject. This restraint is:

$$c_{ij} + K_{ij} \geqq x_{ij} \qquad (2)$$

Eq. 2 states that the original capacity of each route plus the added capacity is at least as great as the traffic moving over the route. The statement holds for all routes. This equation assures that capacity will be added until at least all demands (in terms of flows) are met.

More is required to describe the behavior of traffic in a transportation network. It is known that the traffic on any route is a function of the competitive position of that route *versus* other routes. It is simply noted that when a route is improved, traffic may be diverted from other routes to the improved route. Thus, the improvement of a route may affect flows on other routes. However, there are more complexities to this problem than the simple diversion of traffic. Concurrent with improvements of transportation routes are changes in the competitive positions of urban centers. Some urban centers are now able to reach out over greater areas to market products or compete for raw materials; others have their zones of influence curtailed. The point is that the change of a route from two places, say k to q, may influence the traffic flow into and out of the i^{th} place, and this change may be measured by summing over all outflows of the i^{th} place, namely:

$$\sum_j x_{ij} \leqq \sum_k \sum_q a_{ikq} K_{kq} + A_i \qquad (3)$$

This notes simply that the total outflow from the i^{th} place, $\sum_j x_{ij}$, is less than or equal to the original production at that place, A_i, plus that flow induced by the new capacity on the route from k to q (flow does not exceed capacity). The values of a_{ikq} are coefficients to be determined from empirical experience.

A similar analysis of inflows to the i^{th} place may be made by examining the inflows of products to the place. This restraint may be written in a manner similar to that for the outflow, namely:

$$\sum_i x_{ij} \geqq \sum_k \sum_q b_{jkq} K_{kq} + B_j \qquad (4)$$

Here b_{jqk} is a measure of the influence on the j^{th} place of new route capacity between k and q, and is also to be determined empirically.

The equation to be minimized, the objective function, is also subject to restraints ($K_{ij} \geqq 0$ and $x_{ij} \geqq 0$) to insure meaningful answers.

Algebraic manipulations may be used to arrange the restraining equations so that the variables are restrained by known constants. Thus, the foregoing restraint set may be rewritten as follows:

$$-x_{ij} + K_{ij} \geqq c_{ij} \qquad (5)$$
$$-\sum_j x_{ji} + \sum_k \sum_q a_{ikq} K_{kq} \geqq -A_i \qquad (6)$$
$$\sum_i x_{ij} - \sum_k \sum_q b_{jkq} K_{kq} \geqq B_i \qquad (7)$$
$$K_{ij} \geqq 0 \qquad x_{ij} \geqq 0$$

To make the set of restraint equations as clear as practicable, they may be written in an extended form, as follows:

$$\left.\begin{array}{lll} -x_{12} & +K_{12} & \geqq -c_{12} \\ -x_{13} & +K_{13} & \geqq -c_{13} \\ \vdots & \vdots & \vdots \\ -x_{in} & +K_{in} & \geqq -c_{in} \\ \hline \vdots & \vdots & \vdots \\ -x_{ni} & +K_{ni} & \geqq -c_{ni} \\ \vdots & \vdots & \vdots \\ -x_{n(n-1)} & +K_{n(n-1)} & \geqq -c_{n(n-1)} \end{array}\right\} \text{(Capacity)} \quad (5)$$

$$\left.\begin{array}{l} -\sum_j x_{1j} + \sum_k \sum_q a_{1kq} K_{kq} \geqq -A_1 \\ -\sum_j x_{2j} + \sum_k \sum_q a_{2kq} K_{kq} \geqq -A_2 \\ \vdots \\ -\sum_j x_{nj} + \sum_k \sum_q a_{nkq} K_{kq} \geqq -A_n \end{array}\right\} \text{(Outflows)} \quad (6)$$

$$\left.\begin{array}{l} \sum_i x_{i1} - \sum_k \sum_q b_{1kq} K_{kq} \geqq B_1 \\ \sum_i x_{i2} - \sum_k \sum_q b_{2kq} K_{kq} \geqq B_2 \\ \vdots \\ \sum_i x_{in} - \sum_k \sum_q b_{nkq} K_{kq} \geqq B_n \end{array}\right\} \text{(Inflows)} \quad (7)$$

It is seen from the foregoing that the requirement that capacity be provided for all flows may require as many as $n(n-1)$ restraint equations, and that recognizing the effect of route changes on inflows and outflows of centers requires $2n$ restraint equations. At first glance this may seem to be a system so large as to defy practical analysis. It should be recalled, however, that the $n(n-1)$ accounting of the first group of equations would not hold in practice. Every urban center is not directly connected with every other urban center. An approximation of the number of equations in the practical problem of the interstate highway system indicates $6n$ or $7n$ equations.

The minimum value of the objective function (subject to the restraints) may be found by linear programming methods. Computational routines are available for several large computers and a problem of the size posed by the interstate highway network could be handled by existing methods. Theoretical, computational, and economic aspects of linear programming are reviewed by Dorfman (*4*).

The values of K_{ij} selected for the minimum value of the objective function and their associated values of P_{ij} would indicate the total budget needed to align capacity of the road network with traffic (values of x_{ij}). The transportation costs in the developed system are also indicated by the objective function $(\sum_i \sum_j r_{ij}x_{ij})$.

Suppose the total available budget is less than that indicated by the minimum value of the objective function. How should capacity improvements be allocated in this case? Would the order of selection of capacity improvements affect the realization of the allocation indicated by the objective function? It should be clear that the objective function examined does not contain answers to all aspects of the allocation problems. Questions such as those propounded are not answered by the present formulation. They are the proper subject of further research.

The Dual or Pricing Problem

The solution of the foregoing problem in highway transportation implicitly places values on the various inputs and outputs involved in the system. These values are found in a dual problem to the problem just discussed (the primal). Before identifying these values it will be useful to examine general relationships between primal and dual problems.

The relation between the minimization problem just discussed and the linear programming dual may be conveniently illustrated using general matrix and vector notation. The problem previously discussed was to find a set of values for the elements of a column vector V which would minimize the product, $\Delta_{min} = R'V$, where R is a column vector of known coefficients. This minimization is subject to $BV \geqq X$, where B is a matrix of known coefficients and X is a column vector of known constants.

The dual problem is to find the values of the elements of a column vector U which will maximize the sum

$$\epsilon_{max} = X'U$$

subject to the restraint set

$$B'U \leqq R$$

Formal results from the algebra of linear inequalities indicate that if a finite solution exists for the primal problem, then a finite, and identical, solution exists for the dual; that is, for Δ_{min} and ϵ_{max} finite, $\epsilon_{max} = \Delta_{min}$.

Relationships between dual and primal problems important in the present analysis may now be listed, as follows:

1. The dual of a minimization problem is a maximization problem, and *vice versa*.
2. The dual problem has one restraining inequality for each variable in the primal problem and one variable for each restraining inequality in the primal problem.
3. The inequalities in the dual restraint set have the opposite direction to those in the primal restraint set.
4. The coefficients of the primal objective function appear as restraining constants in the dual restraint set, and the restraining constants of the primal problem are the coefficients of the dual objective function.

To obtain the dual statement of the highway network problem it is necessary to define a dual vector U. Once the primal problem is solved empirically, the solution of the dual problem and values for the elements of the vector may be found by simple mathematical manipulation. Following the definition of dual-primal relationships just elaborated, the dual vector may contain up to

$n(n-1)+2n$ elements. Let this vector be:

$$U = (u_{12}, u_{13}, \ldots, u_{1n};\ u_{21}, u_{23}, \ldots u_{2n};\ \ldots;\ u_{n1}, u_{n2}, \ldots, u_{n(n-1)};\ v_1, v_2, \ldots, v_n;\ w_1, w_2, \ldots, w_n)$$

Multiplication of U against the column of restraining constants of the array of restraint equations of the primal problem yields the objective function of the dual problem:

$$\text{maximize } \epsilon = -\sum_i \sum_j u_{ij}c_{ij} - \sum_i v_i A_i + \sum_j w_j B_j \quad (8)$$

The restraint equations for this problem are found in a similar manner and may be arrayed as the restraint equations of the primal problem were arrayed:

$$\left.\begin{array}{l} -u_{12} - v_1 + w_2 \leq r_{12} \\ -u_{13} - v_1 + w_3 \leq r_{13} \\ \quad \vdots \\ \hline \quad \vdots \\ \hline \quad \vdots \\ -u_{n(n-1)} - v_n + w_{(n-1)} \leq r_{n(n-1)} \end{array}\right\} \quad (9)$$

$$\left.\begin{array}{l} u_{12} + \sum_i a_{i12} v_i - \sum_i b_{i12} w_i \leq p_{12} \\ \quad \vdots \\ u_{n(n-1)} + \sum_i a_{i(n-1)} v_i - \sum_i b_{i(n-1)n} w_i \\ \qquad \leq p_{n(n-1)} \end{array}\right\} \quad (10)$$

The restraint equations for this problem are greater in number than those for the primal problem; there are a possible $n(n-1)+n(n-1)$ restraints in the dual problem. However, it should be recalled that in practical problems many of the possible equations will not appear. The restraining constants are the transportation costs on each route (r_{ij}) and the cost of new capacity on each route (p_{ij}).

Formal mathematical considerations controlled the writing out of the dual problem. Economic, geographic, and engineering considerations control the interpretation of the dual. The specific problem here is that of finding a meaning for the vector U.

By scanning the equation to be maximized (Eq. 8) and the restraint equations (Eqs. 9 and 10), it is seen that the values of u_{ij} are associated with the original capacity and specific routes. The v_i's are associated with movements from urban centers and production considerations (the A_i, and the a_{ijq}). The w_j's are associated with consumption or import considerations. Elements of the vector U are in monetary terms, as is seen by inspecting any one of the equations.

The interpretation of the vector by the authors is that each u_{ij} represents the toll charge for a particular route or a price of the capacity of the route. Each u_{ij} represents a price or a toll which may be imposed for use of the route to ensure that the entire network system is used optimally; that is to minimize cost of transportation in the system. Each v_i is the f.o.b. price of a unit of production at place i, and each w_j refers to delivered unit price at j. The meaning of these prices is clarified by reference to the restraint equations and the objective function.

Restraint Eqs. 9 are constrained by transportation costs, so that for any route the spread between the delivered unit price and the f.o.b. unit price of goods plus the road toll must be equal to unit transportation cost (equalities hold for variables that enter the objective function). This is noting that in the optimum solution (that solution with a maximum value of the objective function) marginal benefit (left side of the equation) equals costs (right-hand side of the equation). Restraint Eqs. 10 indicate that the value of the computed change in production minus the imputed value of increased deliveries (which may be thought of as a site benefit) plus the value of capacity (u_{ij}) must not exceed the price of installing capacity, p_{ij}. The equations have a cost-benefit meaning similar to that of Eqs. 9.

Referring to the dual objective function, it is seen that the difference between production and transportation plus production cost is maximized. Reference here is to the original production. It is interesting to compare this with the primal objective function. The allocation that minimizes the joint cost of new capacity and transportation costs also has a maximizing result in terms of the original layout of the network of producing and consuming centers and routes.

What does this ready availability of the dual mean in the present problem? The answer seems to be roughly as follows: If a selection is made among possible budgets that might be allocated to the road system and a level of allocation is selected, as well as those places of allocation indicated by the primal problem, the dual may be examined for measures of the values of this allocation. Elements of the dual vector establish the value of the capacity increase, u_{ij}, at the road and at the places served by the roads. In the latter case a value is placed on changes in production and consumption levels. At the practical level these dual values might be of great value in the establishment and implementation of taxation policy and in like problems where prices are needed.

EVALUATION

The discussion thus far has brought together features of transportation networks (asserted pragmatically) with a formal analytic scheme (linear programming). The purpose was to describe features of transportation networks, especially from the point of view of expansion of networks. The evaluation of how good the resulting description is may be made in two ways. First, the model may be subjected to empirical testing. Second, the model may be evaluated on a conceptual-logical level. The latter alternate is adopted in the ensuing discussion. However, areas where preliminary empirical testing should be used to evaluate the structure of the model are pointed out.

The evaluation of the study utilizes three questions, as follows:

1. How well does the linear programming formulation fit the type of problem under study?
2. How well does the asserted list of significant characteristics of the transportation network incorporate actual significant variables?
3. How well are the asserted significant characteristics of transportation systems brought into the analysis model?

Applicability of Linear Programming

Evidence available from other studies indicates that linear programming is well suited to the general type of problem here under consideration. This statement is supported by empirical evidence in that many practical problems, such as those discussed by Dorfman (*4*), have been solved using linear programming methods. The general problem of achieving optimal output under a set of restraints and of placing prices on inputs is one to which linear programming applies directly. The problem of investment in the transportation system would seem exactly to fit this problem type. Moreover, as pointed out by Orr (*13*), linear programming links problems of a transportation type to problems of spatial price equilibrium and also to the broad field of location theory. Beckman (*1*) and Garrison (*6*) have also stressed the logical links between transportation problems and location theory (theory which deals with determination of rents, uses of sites, and the movements of goods). However, except for these several examples, and to a limited extent that of Troxel (*15*), there previously has been little or no effort to bring the resource of location theory to bear on problems of a transportation type.

The kind of programming used here is "linear" programming. Do the linearity assumptions of linear programming restrict the suitability of the tool? Technically, the linearity assumptions restrict the problem statement to objective functions and restraint equations which are linear in form. This is a disadvantage, but the disadvantage is offset by the advantage that techniques are available for the efficient computation of problems of a linear nature, whereas the computation of nonlinear problems presents formidable difficulties. At the present stage in model construction it seems urgent to maintain the computational simplicity of the linearity assumptions to facilitate the checking of the model with data. Whether or not in the long run the linearity assumptions are unduly restraining can only be determined by empirical experience.

Significant Characteristics of Transportation Networks

The question might properly be raised as to whether the discussion has included all the important features of transportation networks. This is a question not currently susceptible to specific answer. In a broad way, of course, what is taken to be a significant feature of the network depends on the purpose for which the network is analyzed. At the level of the actual analysis what is significant can be determined only by empirical evaluation of the facts. With one exception, the problem of the validity of the identification of important variables in the transportation system is set aside until experience can be obtained with actual problems.

It will be recalled that in the present study the convention was adopted that direct routes pass between nearest neighbors only. Where urban centers intervene between locations, direct routes are not recognized, routings are *via* intervening urban centers. This convention was adopted to facilitate the analysis of congestion. Insofar as the between-cities portion of the interstate highway system is concerned, this would not seem to be unrealistic formulation.

Consider, however, the transportation network within a city. Connect all origins and destinations with directed arcs. Many intersections — up to $n(n-1)$ — may occur in the route grid and dealing with these intersections is a major problem of the analysis. This is the manner in which researchers like Mayer (*10*) have approached urban problems. These intersections are congestion points to which it would be essential to attach costs and to incorporate systems of restraints. However, it is not immediately clear how these costs could be introduced into the formal analysis.

Failure of the proposed model to recognize congestion costs at route intersections restricts its application to rural portions of the interstate highway system. Urban portions of the interstate highway system, as well as other urban transportation routes (for example, subways) present evaluation problems. How should the theoretical scheme be modified to incorporate such important features of urban transportation systems?

The restraint system of the present model might be enlarged to include the cost-scale features of urban carriers. For example, it could be noted that walking will accommodate a trip by a single individual for a short distance, whereas, subways would accommodate many individuals traveling long distances. The objective function (Eq. 1) would then include the minimization of walking, undesirable length trips *via* the subway, and other pertinent features of the system.

It is noted, however, that these additions to the system have not achieved the desired objective. Congestion costs resulting from actual arrangement of routes have not been recognized. Some work has been done on this problem, but no suggestions currently can be made toward the solution of the problem.

Discussion has been restricted to congestion costs at intersections. Congestion may occur on routes other than at intersections and such congestion is implicit in the present model. Note that one set of restraints in the primal formulation requires that capacity be enlarged to accommodate flows. The act of choosing new capacity indicates that flows are congested (flows exceed capacity). At the end of the allocation period there will be no congestion because restraint Eq. 2 will be satisfied.

In the model, transportation costs (r_{ij}) are assumed to be constant. Observations in the preceding paragraph indicate that transportation costs could not be constant, they would decrease as capacity is added to meet flow requirements. Proper treatment of transportation costs as a function of congestion along routes requires analysis beyond that of the present paper. Beckman (*1*) has made an important start on this topic.

NEED FOR EMPIRICAL RESEARCH

Lack of explicit empirical references limits to generalities the discussion of the suitability of the research model to actual problems. Empirical research is needed to clarify theoretical aspects of the model about which decisive statements can not now be made. Also, empirical research is needed to implement the model. It was assumed through the discussion that information on imputed imports and exports at urban centers following increases in route capacities were known, and that the transportation system was describable in terms of flows, capacity costs, and the like.

At the empirical level many of these bits of information are simply not available. Insofar as the model represents a valid underlying theoretical orientation for studies and is the goal of empirical studies (supplying data for the model), it would be worthwhile to undertake large-scale empirical investigations tailored to the model.

Verifying the Structure of the Model

Empirical research needed to verify the structure of the model should follow six lines of investigation, as follows:

1. Patterns of routes. In the establishment of the model it was asserted that the model approximated the interstate highway system. It is not known to what extent this assertion is true. One thing that is needed is more route information in terms of actual intercity traffic flows. Reference here is to flow information similar to, but more extensive than, that developed by Ullman (*16*) for railroad routes.

The problem of indirect routing is also in need of continued study. It was noted that flows were counted only between adjoining cities. In each case some of the traffic between intervening cities would not terminate or originate in the cities identified. To what extent is this true? The authors are of the opinion that through traffic is a relatively small proportion of all highway traffic, but realize that this assertion is of limited validity and will vary from route to route. Empirical research would clarify this situation. A recent paper by Orden (*12*) indicates a promising means of attacking this problem; namely, a method for the explicit recognition of transhipment problems.

2. Clarification of demand and supply functions for urban centers. In the present study the demand for imports at nodal places is assumed to be a linear function of capacity of routes elsewhere in the system; a similar assumption is made for the output of traffic from urban centers. One question to be answered here on the empirical level is whether or not linearity assumptions hold. Another question relates to the problem discussed under the previous item. Any route capacity change is controlled by Eq. 2; it is a function of the movement over that route, which as already noted, is composed of both through or indirectly routed traffic, and traffic originating and terminating on the single route. Some empirical work would clarify how demand and supply in individual centers and their functions are related to these aspects of traffic.

3. The general linearity of the model. The question of the linearity of the model is a point earlier noted; being dealt with here is the linear programming case of more general mathematical programming. It is not clear at this juncture how the linearity assumptions of the model distort actual occurrences. Some

empirical research on the subject would certainly clarify this question.

4. Functional relationships between prices of capacity increments and capacity increments. One essential feature of the model is that the capacity of routes is increased and some costs are incurred. It would seem reasonable that in an actual situation high threshold costs of increasing capacity would obtain. It is imagined that the present highway system has already had capacity increments in all places where a reasonable cost level exists. Additional increments require such drastic action as the addition of new lanes to routes, creating limited access facilities, and the like.

It would be interesting to know what part threshold costs play in the price of increased capacity.

5. Clarification of multiple-commodity and passenger-*versus*-commodity movements through empirical references. In the model the appearance of multiple commodities was implicit. It was possible to avoid cumbersome notation by carrying the discussion without direct reference to commodities. On a practical level it is not immediately clear how the discussion might be broken down into commodity types. Exploratory empirical research is needed to identify the commodity classes which it would be important to recognize, and to resolve such commodity classifications as might be adopted with industry classifications, where input and output data are available.

Another point is that roads are also used for passenger traffic — the movement of persons. The framework of measurement here is somewhat different from that of the commodity case and some exploratory empirical research would be needed to merge the two.

6. Recognition of effects over time at the empirical level. It is presumed that propensity to travel is changing over time due to pervasive social effects, apart from the present model. The reference here is to passenger transportation. It might be noted that increased use of advertising media for marketing, and the like, is probably increasing the capacity to consume goods from great distances, completely independent of usual cost considerations. Some of these effects need to be identified in order that capacity increases may be realistically programmed over spans of time.

To summarize, it is seen that changes in the system occur internal to the model and that effects occur external to the model. It is necessary to know both classes of effects if the model is to represent the real situation.

Operation of the Model

Three lines of research work can be suggested to prepare for calculations using the model as follows:

1. The a_{ikl}'s and the b_{jkl}'s must be determined. For each commodity and each route one needs to know how capacity changes resulting in traffic flows affect the propensity of urban places to consume and supply. Powerful methods are currently available for the estimation of supply and demand equations, but use of these methods has been somewhat different than that required to supply the data for the present model. Considerations of the type identified here need to be entered into estimations of demand and supply and made available to the present problem.

2. The r_{ij}'s should be determined. It is presumed that costs of transportation are known. This is true, of course, of tariff structures for cartage by commercial carriers, but even here data are not readily available in the form needed for the present study. Cartage by non-commercial carriers (company-owned vehicles) and transportation by private individuals present problems. Research is needed to produce the needed information.

3. The p_{ij}'s need to be determined. It was noted earlier capacity prices are needed at the level of the empirical operation of the model. These will have to be determined. It is presumed that information is already available in the construc-

tion experience of highway operating units. The problem is to make this information available.

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DISCUSSION

WILLIAM S. PETERS, *Montana State University.*— The presuppositions upon which this paper is based are valid; namely, that many highway studies, especially in the cost-benefit area, are fragmentary in character. Not that this is undesirable necessarily, but the theoretical formulation of the problem of which they are a part is often treated in a perfunctory manner. Even when this is spelled out, different investigators will proceed, as is clearly their right, from different theoretical backgrounds. The net result is that it becomes difficult, if not impossible, for yet another investigator to use these heterogeneous studies as a basis for formulating or testing broader types of theoretical statements. Therefore, the present effort is entirely in order. There is a need for this kind of work.

In addition, this paper is a substantial contribution to the theoretical background necessary for a sound over-view of the problem of allocating highway investment. It deserves study, comment, refinements, and extension by others in the field. The theoretical formulation presented is both a "model" which states the interrelationship between basic location and transportation variables and a "model" which can be used more or less directly in cost-benefit approaches to the highway taxation problem. The approach employed can be extended in both directions — backward toward ultimate determinants of traffic generation, and forward toward the highway planning stage and assessment of costs to direct beneficiaries. It would be desirable for both the authors and others to attempt this. The more specific commentary to come may amplify these points and will, it is hoped, begin the process of comment, refinement, and extension which the authors' efforts merit.

In the section titled "Content of a General Statement" the authors state their own list of the requirements of a model. This same framework has been used for organizing the writer's comments, as follows:

1. The supply and demand parameters within which the economic system operates. These are, apparently, the A's and B's (outflows from sources and inflows to destinations) which must be determined empirically for use in the model. Restraint Eqs. 3 and 4 recognize that these flows are affected by investments on routes of which these sources and destinations are not a part, and this is a valuable part of the statement of inter-relationships. But what determines the level of the outflows and inflows? Obviously it is some combination of the location of population and resources plus the configuration of the transport system. Studies pushed in this direction (backwards) aimed at parameters which determine the A's and B's are needed, it would seem to fulfill the stated requirement. Indeed, if such parameters are developed, and they involve forecastable economic variables, the present analysis can be used in the sense of future as well as present planning.

2. Pertinent costs of transportation and transportation facilities. These are involved in the objective function which minimizes the joint or sum of the costs of present transportation plus the cost of new capacity (subject to the requirements of the restraint equations). The joint nature of this objective function is confusing, although it is not apparent if more is involved than a "quibble" over the way these costs are defined. The two costs seem to be of a different character — existing transportation is a current cost, whereas capacity costs are an investment. Would it make sense to minimize annual investment charges in relation to some measure of transportation output of the system? In other words, to drop (explicitly at least) current users' costs and to seek to minimize investment in relation to transportation output, or to maximize output for some fixed investment? There are some variants of the model presented in which the relevant costs may be not the rates of carriers for different commodities but the operating costs of carriers in different vehicle classes. More is given on this point later.

3. Points of production and routes of marketing for all commodities or services which use transportation. The "nodal points" concept yields the spatially separated centers of production and consumption — or termini of a segmented network. But to recognize routes of marketing, as the writer understands the term, would require the recognition of through routes, not the segmenting of flows by directly connected "nodal points." There are good reasons for the flow segmentation used and it does make possible certain kinds of applications which would otherwise seem extremely difficult. However, the present formulation does not permit of a solution calling for new investment to link directly two locations not previously joined. It would seem that if "marketing routes" were identified in terms of their originating and terminating points, as well as the connecting segments, this possibility would exist. Without this, the solution always will be in terms of increasing the capacity of the existing network; not to create entirely new capacity where it may be justified by the origin-destination pattern of physical distribution in the economy.

4. Geographic realities of the orientation of routes and locations of nodal

points. Insofar as this refers to the directional aspect of routes introducted by the authors, the writer was inclined at first to agree with this "departure." Some complications may ensue, however, for this leads to a possible solution in which capacity is to be increased from i to j but not from j to i. Is this possible? Is it practical in view of construction practices? If the effect is inevitably to increase capacity from j to i also, how can the investment be wholly charged to route ij? To the extent that capacity increases are in fact joint costs of movement in both directions, the route from i to j is one "commodity" with that from j to i, despite the fact that the movements in either direction are not equalized. This seems a "messy" problem: possibly the movement to be taken into account as far as the need for capacity is concerned is that in the direction of greatest movement. The model essentially does just this, but the reverse movement may enter into the benefit of increased capacity.

5. Changes with time. The linear programming formulation is technically a static equilibrium model. Change over time is introduced by appropriately changed values of the knowns in the equations. The model does not explain how these changes come about, nor does it consider the process whereby a new equilibrium is reached. Truly dynamic models are hard to come by: the present static model would have more dynamic overtones were the writer's previous suggestion to press back toward ultimate determinants of traffic generation taken up.

This general review has anticipated some of the comments concerning the formulation of the model. Additionally, the following are offered for consideration:

1. *Definition of route capacity.* This is of course critical to any application of the model. Apparently it is some fixed number which must for some routes in the system be less than current movement. Problems in devising an operational definition of capacity are anticipated. As a concept, capacity seems to be a flexible term, a function of the transportation and congestion costs which could be tolerated on any given route. It would require quite a reshaping of the analysis to replace the capacity term with some measure of route potential, the problem being to allocate additional investment among routes in some optimum fashion to bring actual movement patterns more in line with potentials. And such a move might not improve the matter any.

2. *Inflows and outflows* (*restraint Eqs. 3 and 4*). The incorporation of the influences of highway investment in given routes on flows along other routes is certainly a theoretical and practical necessity. The writer is somewhat pessimistic about the possibility of empirical studies providing the critical coefficients (the *a*'s and the *b*'s). There is need for a supplementary theory or model which will allow these to be estimated from other more readily accessible variables. Historical examples may provide partial tests of such models, but the required coefficients are of such a net character that analysts may have to be satisfied with a model for estimating them that simply meets certain logical tests.

3. *Addition of the commodity dimension.* The authors mention that some study is necessary to determine the kinds of commodity classifications most appropriate to the analysis. However, the model may be susceptible to a wholly different view of "commodities." The *x*'s in this application would represent transport demand over routes by vehicle types and the *r*'s the operating costs of vehicle types over various route segments. The problem would be to determine annual investment charges to routes so as to minimize total operating plus investment charges. This extension would move the analysis a step closer to the highway taxation problem. Demands by vehicle type are derived from underlying commodity flows, but traffic data by kind of vehicle and operating cost data by vehicle type for various conditions of terrain,

traffic density, and highway types would seem to be more readily available. The concept of segmented routes seems to fit r's defined as operating costs better than r's viewed as carriers' rates, for in the latter case there will be a problem in breaking up through rates by route segments. The dual solution, which has great significance to taxation problems, would seem in this modification to yield charges required to recover investment expenditures *via* user taxes.

4. *Sub-optimization.* As the authors point out, the present model represents sub-optimization with respect to investments in the entire transportation system. A model cannot be expected to solve all problems, but investment in other means of transport will affect highway flows. Where these are known and substantial, some effort might be made to modify the x's (route flows) directly.

The method of segmenting the transportation network seems to facilitate regional and sub-regional applications of the technique. It would be possible to use the model to allocate a predetermined total investment budget among inter-regional links, or a total regional budget among intra-regional routes. Or the model could be applied to previously recognized sub-systems, such as federal interstate, federal aid primary, federal aid secondary and so forth. Thus, it is flexible enough to operate in a situation in which some budget decisions are politically but not economically optimal, whereas in others economic optimization can realistically be sought.

The authors did a fine job of pointing out the implications and limitations of their own work. The writer was impressed and intrigued, for instance, by their elucidation of the economic and fiscal implications of the dual solution. This is a powerful device for obtaining insights not otherwise easily come by. It will be interesting to see how these work out in various modifications and extensions of the basic formulation.