Some Characteristics of the Modulus of Passive Resistance of Soil: A Study in Similitude

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Approximately 20 years ago the Iowa Engineering Experiment Station, in cooperation with the Bureau of Public Roads, conducted research directed toward the rational design of flexible pipe culverts under earth fills. Combined analytical and experimental studies of this problem resulted in the development of the Iowa formula for predicting the deflection of a structure of this kind.

Full-scale pipelines ranging in diameter from 36 to 60 in. under fills of several kinds of soil in several different states of compaction were used in the early experiments. From these studies, the modulus of passive resistance appeared to be a function of the type of soil and its density or state of compaction. They did not reveal any influence of other components of the pipe-fill system, such as diameter of pipe, height of fill, and stiffness of the pipe wall. It was therefore assumed that this modulus was a function of soil properties alone.

Subsequent attempts to apply the Iowa formula to actual flexible pipe installations have led to confusing results, which appear to be due in large measure to a lack of understanding of the true character of the modulus of passive resistance of soil.

Against this background of research and experience, an attempt has been made to delve deeper into the nature of the modulus, employing the principles of engineering similitude and the Buckingham pi-theorem in a study of the problem. One result of this study is the qualitative showing that the modulus of passive resistance is not a function of the soil alone, but is materially influenced by the diameter of the pipe. Specifically it is shown that the modulus e is not a constant for a given type of soil, but rather that the product $e\ r$ is constant, where r equals the radius of the pipe. This opens up a whole new avenue of approach to the problem of rational design of flexible pipe culverts and gives promise of aiding in the clarification of some uncertainties which have persisted since the Iowa formula was first published.

• DURING CONSTRUCTION of a soil fill over a flexible pipe culvert the vertical diameter of the pipe decreases and the horizontal diameter increases, causing the pipe to bear laterally with increasing force against the adjacent soil. The greater the lateral bearing resistance of the soil, the less will be the deformation of the pipe, and the less will be the chance of failure. It has been demonstrated from extensive field practice that failure of a flexible pipe culvert is conveniently expressed in terms of excessive deformation of the pipe (*9*, p. 340). If the vertical diameter decreases by about 20 percent from the initial circular diameter, the pipe is said to be in a state of incipient collapse. Additional vertical load on the pipe will cause failure (*1*, p. 70). It is customary, then, to refer to failure conditions in a flexible pipe culvert as 20 percent decrease in

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vertical diameter. Allowable values are commonly accepted as 5 percent decrease in vertical diameter or 5 percent increase in horizontal diameter.

In order to design flexible pipe culverts according to this basis, Spangler (*8*, p. 29) derived an equation for predicting the increase in horizontal diameter, Δx, for a pipe embedded in a soil fill. This equation, often referred to as the Iowa formula, is:

$$\Delta x = \frac{K\,W_c r^3}{E\,I + 0.061\,(er)\,r^3} \tag{1}$$

in which

- Δx = increase in horizontal diameter of the pipe culvert (in.);
- K = a parameter which is a function of the pipe bedding angle, α;
- W_c = vertical load per unit length of the pipe at the level of the top of the pipe (lb/in.);
- r = mean radius of the pipe (in.);
- E = modulus of elasticity of the material from which the pipe is constructed (psi);
- I = moment of inertia of the cross-section of the pipe wall per unit length (in.^{4}/in.); and
- e = modulus of passive resistance (lb/in.^{2}/in.).

All of these factors are readily determinable for a proposed flexible pipe installation except the modulus of passive resistance, e. The exaluation of this modulus is a major stumbling block to the use of the Iowa formula. The modulus is a measure of the lateral bearing resistance or support contributed by the adjacent soil as the sides of the pipe move outward. It is similar to the Westergaard (*12*) modulus of subgrade reaction and Cummings' (*2*) modulus of foundation in that it is a measure of the rate of change of lateral pressure with respect to lateral displacement. Terzaghi (*10, 11*) has done some work with a quantity defined in the same way as e, which he designates as the coefficient of horizontal subgrade reaction. He uses it in predicting the horizontal load capacity of a pile.

Written mathematically,

$$e = \frac{2dh}{d(\Delta x)} \tag{2}$$

in which

- h = maximum horizontal soil pressure against the pipe (assumed to act at the extremity of the horizontal diameter); and
- $\frac{\Delta x}{2}$ = horizontal displacement of the pipe at the point on which h acts, and is equal to one-half the horizontal deflection of the pipe.

Eq. 1 shows that the greater the modulus of passive resistance, the less the deflection, Δx, and the less the chance for failure. If e were zero, the pipe culvert would fail by deformation under a relatively small vertical load, which would cause it to collapse for lack of lateral support. On the other hand, if e were very large, the conduit would support a tremendous load up to the condition at which the pipe wall would fail by crushing or buckling. Inasmuch as modulus of passive resistance, e, describes this range of values, it is a very influential factor in the safe and economical design of flexible conduits. Unfortunately it is ignored in most present-day design. So little is known about it that designers customarily resort to over-simplified experience tables with a factor of safety of four or more. For example in the Armco "Handbook of Drainage and Construction Products" (*1*, pp. 105-143), design tables for flexible steel pipe culverts specify a type of corrugation and a gage number (thickness of metal) for a given diameter of flexible steel pipe as a function of height of fill. Each gage number listed is based on Shafer's empirical equation (*6*, p. 355)

$$y_c = k\,\frac{H^m D^n}{t^s} \tag{3}$$

in which

- y_c = vertical decrease in diameter;
- k = a constant;
- H = height of fill;
- D = diameter of pipe;

t = thickness of the metal (gage); and
m, n, s = exponents.

The exponents and the constant k in Eq. 3 have been evaluated for an average of numerous pipe installations at various deflections regardless of the type of fill or the compaction. The Armco tables are based on a 5 percent deflection, which represents a safety factor of 4 because it is assumed that 20 percent change in the vertical diameter is the definition of incipient failure (*1*, p. 70). No attempt is made whatsoever to consider the effect of the modulus of passive resistance on the gage of metal to be specified for a given pipe.

Such design practices need alteration because, despite the opportunity for overdesign (with its related lack of economy), designers still suffer the embarrassment and expense of occasional failures. Moreover, such design practices offer no incentive for care in compaction or selection of fill, although the handbook referred to emphasizes the need for compaction of the side fill material.

In this paper the principles of engineering similitude have been employed in an attempt to shed more light on the true nature of the modulus of passive resistance and its influence on the deflection of flexible pipe culverts.

HISTORICAL BACKGROUND

Eq. 1 was first published in 1941 (*8*). In connection with the derivation, Spangler reported the results of three series of experiments designed to test the applicability of the equation. In general, the experiments showed that Eq. 1 may be used successfully for predicting the deflection of flexible pipe culverts if an appropriate value can be found for the modulus of passive resistance of the soil. Some of the conclusions resulting from these early experiments are summarized as follows:

1. Qualitatively, the greater the density of the soil adjacent to the pipe, the greater the modulus of passive resistance. It was found that even with compaction much below standard Proctor density the value for e was twice as great as for uncompacted fill (*8*, p. 65).
2. The modulus of passive resistance appears to be independent of the height of fill, both for uncompacted and compacted fill. (*8*, p. 24 and *9*, p. 343).
3. "The Modulus of passive resistance, e, appears to be a function of the properties of the soil . . ." (*8*, p. 5); that is, e is a constant for any given set of soil characteristics.

In 1948 a summary of Spangler's work on flexible pipe culverts was included in a paper (*9*) presented to the American Society of Civil Engineers. The paper drew discussion from a number of engineers who have had experience in culvert design. Shafer (*6*, p. 357) recognized that the weakness of Eq. 1 is in the evaluation of e. He further attempted to verify the equation by solving for height of fill, H, in terms of gage of metal, t, and diameter of pipe, D, using both Spangler's Eq. 1 and his own empirical equation. His table of comparisons showed that some values for height of fill, H, agree, but that many other values are in serious disagreement. Some discrepancy is inevitable, as the empirical equation does not take into account differences in soil and differences in lateral support provided by the soil sidefills. Nevertheless, general similarity of results should be expected for average conditions at least. Shafer concluded that the Iowa formula could not be verified.

Kelley (*3*, p. 364) arrived independently at a similar conclusion with respect to the validity of Eq. 1. His contribution to the problem will be discussed at greater length later in this paper.

Although recognizing a basic weakness in the applicability of the Iowa formula, Shafer (*6*, p. 357) also recognized a need for rationality when he said: "Those engaged in the manufacture and distribution of flexible drainage structures see a definite need for a rational method of design. Not because design based on experience, such as the empirical equation, is wrong, but because it is possible that the full economy of flexible construction is not utilized in all cases. Furthermore, engineers prefer a rational approach to any problem, even though they use empirical methods in the solution of many problems. The rational approach also leads to a clearer understanding of the basic principles involved."

GENERAL PRINCIPLES OF SIMILITUDE

When a quantity appears to be a function of numerous other quantities, and particularly if the functional relationship with any of the other quantities is not readily understood, it is often advantageous to begin the investigation by applying principles of engineering similitude.

Such an approach appears to have merit in the investigation of the modulus of passive resistance. Not only is e the function of many primary quantities, but there also is considerable disagreement as to just what primary quantities actually affect it. For example, Spangler's experiments (*8*, p. 24) tend to show that e is a constant for a given type of soil in a given state of compaction. They also indicate that it is practically independent of height of fill, H, and of the stiffness factor, EI. On the other hand, Shafer (*6*, p. 360) suggests the possibility that it is a function of both H and EI. By employing principles of similitude, an orderly method may be developed for establishing and defining all such functional relationships.

The theory of similitude is conveniently presented by considering a generalized π-term relationship which describes the performance of any given physical system. Such a relationship is:

$$\pi_1 = F(\pi_2, \pi_3, \pi_4, \ldots \pi_s) \qquad (4)$$

in which π_1 is a function of π_2, π_3, π_4, etc. In this generalized form each π-term is dimensionless, is independent of all other π-terms, and represents one or more of the primary quantities which affect the system. Because Eq. 4 is dimensionless, it is perfectly general and applies to any system which is a function of the same variables, regardless of the units of measurement and regardless of the magnitudes of the measured quantities.

A number of examples of π-term relationships are encountered in engineering practice. A typical case is the Reynold's number in fluid mechanics (*5*), by means of which the size effect or scale effect in the design of hydraulic models can be evaluated with respect to the velocity of the flowing fluid. Reynold's number is a dimensionless quantity expressed by

$$R = \frac{VD\rho}{\mu} \qquad (5)$$

in which the primary quantities are:

R = Reynold's number which defines type of flow, such as critical flow between laminar and turbulent;
V = velocity of flow of the liquid (LT^{-1});
D = diameter of pipe (L);
ρ = mass density of the liquid ($FT^2 L^{-4}$); and
μ = viscosity of the liquid (FTL^{-2}).

Because R is dimensionless, it is applicable to any two or more flow systems regardless of the magnitude and units of the length scale used. Eq. 5 is a π-term relationship, just as is Eq. 4. In order to determine the size effect if the same fluid and the same type of flow is desired in a pipe of different diameter, Eq. 5 may be rewritten with primed values as follows:

$$R' = \frac{V' D' \rho'}{\mu'} \qquad (6)$$

where D' is different from D; that is, $D = nD'$ if n = the length scale factor. It is recognized that Reynold's number is a constant for a given flow type so

$$R = R' \qquad (7)$$

It follows that the right-hand sides of Eq. 5 and 6 must be equal or

$$\frac{V' D' \rho'}{\mu'} = \frac{V D \rho}{\mu} \qquad (8)$$

If the same fluid is used in both systems, $\rho' = \rho$ and $\mu' = \mu$, and it is evident that

$$V' = V \frac{D}{D'} = nV \qquad (9)$$

Eq. 9 is a statement of size effect on the velocity of flow if the diameters of the two pipes are different.

Reviewing the foregoing analysis, size effect was arrived at in two basic steps:

A. The primary quantities influencing the phenomenon were arranged in a dimensionless form called a π-term relationship, such as Eq. 5, or, more generally, Eq. 4. The same π-term relationship applies regardless of scale or magnitude of the system, so it was rewritten for a second system of different size using primed notation.

B. Corresponding π-terms of the two π-term relationships were equated. The resulting equations indicate the size effects on the two systems.

It should be pointed out that size effect is not the only quantity that can be investigated by principles of similitude. For example Eq. 8 could have been used to investigate the effect of varying fluid density or viscosity, just as well as size of pipe, in the two systems.

Similarly, the Froude number is employed to obtain similitude in a model system with respect to the force of gravity; the Euler number with respect to certain boundary influences; and the Weber number with respect to capillary forces.

As previously noted, each π-term is a dimensionless quantity which includes one or more of the primary quantities that affect a given system. This provides a starting point from which a general π-term relationship may be written for any system. In this investigation, it is proposed to study the modulus of passive resistance, e, and to determine the size effect on it, if any. The system consists of a flexible pipe culvert under a high earth fill. It is first necessary to list all possible independent primary quantities which appear to influence e. There is a possibility that one or more of the primary quantities listed may have no effect on e, but this contingency clears itself as the investigation proceeds. A reasonable set of such primary quantities follows:

		Dimensions
1.	e = modulus of passive resistance	$F L^{-3}$
2.	r = mean radius of the pipe	L
3.	λ = any other pertinent length dimension	L
4.	EI = stiffness factor for the wall of the pipe (full length of pipe)	$F L^{-2}$
5.	P_v = vertical soil pressure at any point in the earth fill	$F L^{-2}$
6.	v = void ratio of the soil (a function of density)	
7.	w = water content of the soil (percent)	
8.	ϕ = internal friction angle of the soil	
9.	c = cohesion of the soil	$F L^{-2}$
10.	Ω = compactive effort (work per unit volume)	$F L^{-2}$

The bedding angle, α, which defined the width of the pipe bedding is not listed because it can be written in terms of λ, which also includes such primary quantities as configurations of boundaries, displacement of the boundaries, and displacement of any point in the soil.

The foregoing set of primary quantities can be arranged into a general π-term relationship. This may be done arbitrarily, just so the π-terms are all dimensionless and independent of each other and all of the primary quantities are included. The Buckingham pi-theorem (*4*) states, in substance, that the minimum number of π-terms required is equal to the number of primary quantities minus the number of dimensions in which these quantities are expressed. Since there are 10 primary quantities in this case and since these are expressed in 2 dimensions (F and L), the required number of π-terms is 8.

One possible π-term relationship may be written as follows:

$$\frac{er}{\Omega} = f\left(\frac{\lambda}{r}, \frac{E\,I}{\Omega\, r^4}, \frac{P_v}{\Omega}, v, w, \phi, \frac{c}{\Omega}\right) \quad (10)$$

Suppose a second pipe-fill system with a different pipe radius is to be compared with the original to determine the size effect on e. The second, or primed, π-term relationship becomes

$$\frac{e'r'}{\Omega'} = f\left(\frac{\lambda'}{r'}, \frac{E'\,I'}{\Omega'(r')^4}, \frac{P'_v}{\Omega'}, v', w', \phi', \frac{c'}{\Omega'}\right) \quad (11)$$

where $r' = r/n$.

This completes Step A of the similitude study. Step B is carried out by equating corresponding π-terms of Eqs. 10 and 11. The size effect on e can be seen immediately by equating the π_1-terms or the left sides of Eqs. 10 and 11, but it is important to point out that this can be done only if corresponding π-terms in the right sides of Eqs. 10 and 11 are also equal. These equations become a set of conditions of similitude which must be satisfied if the size effect on e is to be determined. These conditions are as follows:

1. $\lambda' = \dfrac{\lambda}{n}$ This requires that geometrical similarity exist throughout the soil of both systems. This condition applies to bedding angle; all boundaries in the soil such as the pipe boundary, rock ledges, wing walls, and bed rock; and dimensions and positions of zones or strata of differing soil properties.

2. $E'I' = \dfrac{E\,I}{n^4}$ This relates the stiffness factors for the walls of the pipes in both systems.

3. $P'_v = P_v$, $v' = v$, $w' = w$, $\phi' = \phi$, $c' = c$ These conditions indicate that all soil characteristics of both systems, including vertical soil pressure, must be the same.

If these conditions of similitude are all met,

$$\frac{e'r'}{\Omega'} = \frac{er}{\Omega} \quad (12)$$

Now if the soil characteristics in both systems are identical, compaction is the same and $\Omega' = \Omega$. Also $r = n\,r'$ so

$$e'r' = e\,r \text{ and } e' = n\,e \quad (13)$$

This is the required size effect on e.

Apparently the modulus of passive resistance is not a property of the soil alone, for it is shown by this analysis to be not a constant in both pipe-fill systems as soil characteristics were assumed to be. Rather, if conditions of similitude are met, it appears that the product $e\,r$ remains constant in systems which have the same soil characteristics. This would indicate that e varies inversely as the pipe radius.

Preliminary tests on model and prototype pipe-fill systems clearly substantiate the foregoing conclusion that $e\,r$ is constant if the conditions of similitude are met. The question now arises as to the effect on $e\,r$ if conditions of similitude are not met with respect to the requirement that $E'I' = \dfrac{E\,I}{n^4}$ (condition 2). Studies of models designed on the basis of similitude have indicated that the effect of variations in $E\,I$ on $e\,r$ is so small as to be negligible. This was arrived at by actually measuring the deflection, Δx, of model pipes under known loading. The Iowa formula was then resolved for $e\,r$ in terms of Δx and the load for various values of $E\,I$.

Regarding the requirement that soil characteristics be constant, model studies indicate that $e\,r$ is practically independent of P_v (or height of fill) for deflections at least between 2 and 7 percent. As would be expected, $e\,r$ is highly dependent on degree of compaction, Ω. No other soil characteristics have as yet been investigated.

The present conclusion is that for conditions of geometrical similitude, $e\,r$ is practically a constant for any given set of soil characteristics. By recognizing this fact several important questions relative to the rational design of flexible pipe conduits under earth fills can be answered. For example, in 1944 de Capiteau pointed out that use of a constant value of the modulus of passive resistance in Eq. 1 led to results which were difficult to accept. He showed that the height of fill which would produce a deflection equal to a constant percentage of pipe diameter decreased as pipe diameters increased up to some value and then increased with further increase in diameter. This result violated intuitive judgment, but at that time there was not sufficient evidence, either theoretical or experimental, to explain this contradictory phenomenon.

Later Kelley (*3*) raised the same question and published a diagram illustrating the rise in height of fill with increase in diameter beyond a certain value. This diagram is reproduced as Curve A in Figure 1. In preparing this diagram, Kelley used a value of $e = 20$ psi per in. If the 36-in. diameter pipe is arbitrarily selected as being typical of the diameters used in his illustration, $e\,r = 360$ psi.

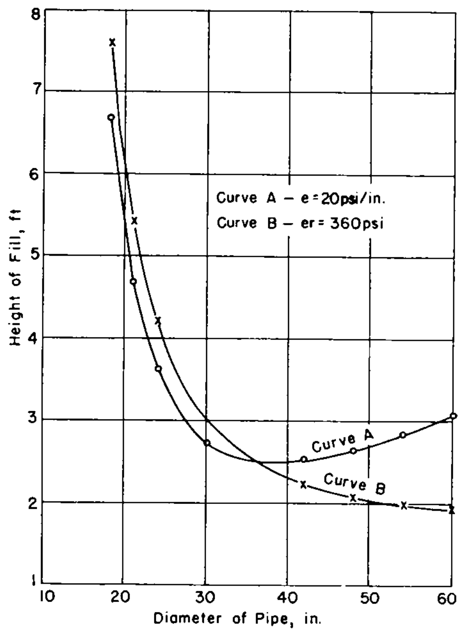

Figure 1. Height of fill *vs* diameter of pipe with *e* constant and with *e r* constant.

In the light of this more recent study, the heights of fill may be calculated using a constant $e\,r = 360$ psi instead of $e = 20$ psi per in. Keeping all other quantities which are involved in Eq. 1 constant and at the same value used by Kelley, the height of fill *versus* pipe diameter is as indicated by Curve B in Figure 1; that is, the height of fill to produce deflections equal to a constant percentage of pipe diameter continues to decrease as diameters increase. This is a much more acceptable result and lends a measure of confidence to the findings of the similitude study reported in this paper.

Barnard (*13*), in a discussion of the Iowa formula and the modulus of passive resistance of soil, stated that he had determined values of e in a number of cases of ordinary steel water pipe installations where deflection was measured and had found e to vary inversely with pipe diameter quite consistently. His observation is in complete harmony with the results of this similitude study. Publication of the data upon which his conclusion is based would provide a valuable physical check on the theoretical results described in this paper.

In the light of the findings of this project it is interesting to determine the values of the product $e\,r$ which have developed in a number of actual culvert pipe installations. In these cases, the values of the modulus of passive resistance were actually measured in some instances and estimated from measured deflections in others. The diameters of the pipes and heights of fill were, of course, known. These values of $e\,r$ are given in Table 1. They show a wide range of variation, from 234 psi in the case of an untamped sandy clay loam soil at the sides of a 36-in. pipe to 7,980 psi for a crushed sandstone soil compacted to Proctor density at the sides of an 84-in. pipe.

This wide range of values reflects the important influence of different kinds of soil in different states of compaction. It is believed that the way has now been opened for determining usable values of $e\,r$ for a range in soil types and degree

TABLE 1

VALUES OF $e\ r$ FOR 17 FLEXIBLE PIPE CULVERTS

Item No.	Location	Pipe Diam. (in.)	Soil Type [1]	Fill Height (ft)	Mod. of Passive Resist., e (psi/in.)	Value of $e\ r$ (psi)
1 [2]	Ames, Iowa	42	Loam top soil (U)	15	14	294
2 [2]	Ames, Iowa	42	Well-graded gravel (U)	16	32	672
3 [2]	Ames, Iowa	36	Sandy clay loam (T)	15	28	502
4 [2]	Ames, Iowa	36	Sandy clay loam (U)	15	13	234
5 [2]	Ames, Iowa	42	Sandy clay loam (T)	15	25	525
6 [2]	Ames, Iowa	42	Sandy clay loam (U)	15	15	315
7 [2]	Ames, Iowa	48	Sandy clay loam (T)	15	29	696
8 [2]	Ames, Iowa	48	Sandy clay loam (U)	15	14	336
9 [2]	Ames, Iowa	60	Sandy clay loam (T)	15	26	780
10 [2]	Ames, Iowa	60	Sandy clay loam (U)	15	12	360
11 [3]	Chapel Hill, N. C.	30	Sand	12	25	375
12 [3]	Chapel Hill, N. C.	31.5	Sand	12	56	882
13 [3]	Chapel Hill, N. C.	30	Sand	12	80	1200
14 [3]	Chapel Hill, N. C.	20	Sand	12	35	350
15 [3]	Chapel Hill, N. C.	21	Sand	12	82	861
16 [3]	Culman Co., Ala.	84	Crushed sandstone (C)	137	190	7980
17 [3]	McDowell Co., N. C.	66	Clayey sandy silt (C)	170	40	1320

[1] U = untamped; T = tamped; C = compacted.
[2] Side pressure and pipe deflections measured.
[3] Side pressures estimated, pipe deflections measured.

of compaction which will greatly facilitate the safe and economical design of flexible pipe culverts under earth fills.

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