Consolidation of Soil Under Time-Dependent Loading and Varying Permeability

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Use of the theory of consolidation in many instances is found to be less than satisfactory due to the working conditions inherent in the usual theory. This paper extends the theory by generalizing the conditions of derivation to include variations in the rate of loading and variations in the coefficient of permeability during the process of consolidation. Including the two additional working conditions, the basic differential equation is derived in its most general form. The resulting differential equation is non-linear, but an exponential approximation is developed to linearize it, thereby permitting solution of the consolidation boundary value problem.

The one-dimensional consolidation problem is treated in detail for a clay-soil layer sandwiched between two sand layers. A general solution is presented for constant permeability and an arbitrary loading rate. Specific solutions are presented for the consolidation during a linear construction period. With these solutions, working curves are presented for actual time computations. In addition, the theory of variable permeability is developed in detail in terms of specific solutions.

The consolidation of a doubly-drained layer of clay-soil under the influence of two- and three-dimensional loading is presented. These solutions are developed in general, and for specific linear loading functions, considering constant permeability. Extensions to problems of variable permeability are indicated. The three-dimensional consolidation problem is also specifically oriented to the radial drainage of a triaxial test specimen under a stress-controlled loading.

Sand-drain problems for equal-strain conditions and radial flow only are covered in detail. The exact solution to consolidation under constant load but variable permeability is presented in terms of the well-known constant permeability solution, which considers a perfect drain without smear. Constant permeability solutions are presented for the case of a linear construction loading, in which the presence of a smear zone is included in the solution. The exponentional linearizing approximation is developed for situations of variable load and permeability for the case of sand drains without smear. A set of working curves is presented for the design of sand-drain installations using these theories.

FORMULATION OF THEORY

• THE MODERN history of soil mechanics is directly traceable to development of the theory of consolidation by Terzaghi (*1*). This theory for the first time developed a procedure based on mathematical rigor, enabling engineers to predict the time rate of settlement of structures constructed on fine-grained soils. In addition, the Terzaghi theory isolated those variables which influence and control the rate of settlement.

The treatment of this subject by Terzaghi and Frohlich (*2*) considered, among other items, two conditions of imposed load in the one-dimensional case. The first condition was that of "creep," wherein the entire foundation load is applied to the soil at a single instant in time. The application of this condition is in common practice today. The second condition, considered by Terzaghi and Frohlich to be a variation on the creep theory, was one in which the imposed loading was increasing at a constant rate, corresponding to a construction cycle. It is current American practice to use the "creep" theory and to account for time-dependent loading by a graphical procedure developed by Taylor (*3*).

The Terzaghi theory for the time-dependent loading is a one-dimensional theory and the graphical procedure is applicable only to vertical flow of water through thick layers of fine-grained soil. This theory, although adequate in its present context, does not extend to such modern developments as sand drains and does not consider cases of nonlinear loading.

One of the fundamental assumptions of the existing theory of consolidation is that the capacity of the soil to transmit water during consolidation is a constant. The assumption that the coefficient of permeability is a constant is generally recognized as an over-simplification of the physical processes involved. The errors involved in this assumption are thought to be smoothed by the use of average values of permeability. Recent studies of the mechanism of permeability by Schmid (*4*), however, have reached a stage where the effect of time-dependent permeability can be included in the formulation of the theoretical equations governing consolidation and in some cases can be included in the solution.

This study develops a general theory for time variations of the imposed loading, and permeability, and considers several problems most likely to be encountered in practice, such as sand drains and two- and three-dimensional loading, as well as vertical consolidation.

Notation

The symbols used throughout this paper are defined in the Appendix for convenience of reference.

Framework of Conditions

The basic framework of conditions on the soil mass and its properties for the general case of variable loading consolidation of fine-grained soils are as follows:

1. The soil mass is completely saturated with an incompressible fluid, and consists of incompressible solids of small particle size. The condition of compressibility of soil solids has been treated by Richart (*5*), and it was shown that this condition leads to a nonlinear differential equation whose numerical solution is so close to the usual solution that this case, as well as others leading to similar situations, will not be considered.

2. Darcy's law of permeability is instantaneously valid. Although it is assumed that the permeability of the soil will vary in time and space, a basic postulation in this theory is that at any instant of time the flow of water will follow Darcy's law. This condition is formulated as follows:

$$\vec{v} = k\,\nabla h \tag{1}$$

The velocity, $\vec{v}$, in this theory is considered to be a vector point function which at any given point in the mass will vary with time. As a result the coefficient of permeability, k, is a scaler point function and will vary from instant to instant. This condition forms the basis for development of a three-dimensional anisotropic theory of consolidation.

3. The change in volume with imposed pressure is linear and is small as compared to the original volume. These postulations permit development of a linear small-strain theory. Other postulations, which may be closer to reality, would introduce non-linear differential equations and increase the tediousness of solution

out of proportion to the increases in accuracy.

Fundamental Equations

Consideration of the flow of fluid through the fine-grained porous media is based on the concept of the law of conservation of mass.

Given a fixed closed volume within the porous mass with volume V and surface S, as shown in Figure 1, the fluid will flow into the mass through the surface at the instantaneous rate:

$$\int_s \vec{v}\cdot\vec{n}\, dS \tag{2}$$

The increase in volume due to a head of fluid being generated internally at rate Q will be:

$$\int_v Q\, dV \tag{3}$$

The total instantaneous mass flow through the surface S will be:

$$\int_s \vec{v}\cdot\vec{n}\, dS + \int_v Q\, dV \tag{4}$$

By Gauss's theorem, Eq. 4 can be transformed to

$$\int_v (\nabla\cdot\vec{v} + Q)\, dV \tag{5}$$

Substituting Darcy's law in Eq. 5 gives

$$\int_v [\nabla\cdot(k\nabla h) + Q]\, dV \tag{6}$$

The porous mass in question will undergo an elementary volume change due

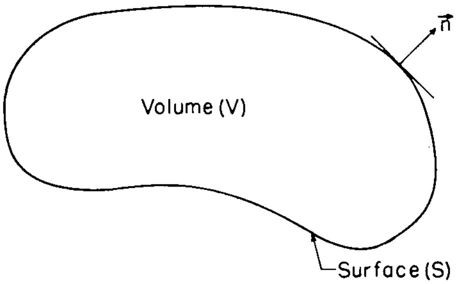

Figure 1. Arbitrary closed volume within soil mass.

to the flow of water along the given flow path.

$$\frac{\partial V}{\partial t} = \frac{\partial V_v}{\partial t} + \frac{\partial V_s}{\partial t} \tag{7}$$

By the condition of incompressibility of the soil solids the second term of Eq. 7 will vanish, thus

$$\frac{\partial V}{\partial t} = \frac{\partial V_v}{\partial t} \tag{8}$$

The total volume change in the arbitrary volume V will then be

$$\int_v \frac{\partial V_v}{\partial t}\, dV \tag{9}$$

By the law of conservation of mass, the rate of mass flow must be equivalent to the total volume change, and

$$\int_v \left[\nabla\cdot(k\nabla h) + Q + \frac{\partial V_v}{\partial t}\right] dV = 0 \tag{10}$$

Inasmuch as V is an arbitrary volume,

$$\nabla\cdot(k\nabla h) + Q + \frac{\partial V_v}{\partial t} = 0 \tag{11}$$

The fluid in question is water and the relation between the head and the excess pore pressure is:

$$u = \gamma_w h \tag{12}$$

The classical theory of consolidation, as well as this theory, considers strains due exclusively to volume changes; thus, only normal stresses are considered. In the theory of elasticity (*6*), the ratio between the mean stress and the dilitation is defined as the "modulus of compression." In this theory the ratio between the change in volume and the mean stress, Θ, will be termed the "modulus of volume change," m.

$$m = \frac{\delta V_v}{\delta\Theta} \tag{13}$$

in which

$$\Theta = \sigma_x + \sigma_y + \sigma_z \tag{14}$$

A more complete discussion of this topic has been made by Biot (*7*) and is beyond the scope of this study.

The established concept of neutral and effective stresses states that within a saturated soil mass the total applied stress across any surface is made up of the neutral stress, u, and the effective mean stress, $\bar{\Theta}$.

$$\Theta = \bar{\Theta} + u \qquad (15)$$

The physics of consolidation, as formulated by Terzaghi (*1*), is based on the principle that initially the total applied stress is developed as neutral stress (excess hydrostatic pressure) and that as flow develops, relieving the excess hydrostatic pressure, the applied stress becomes effective. The change in the effective stress, furthermore, follows the same law as the volume change. Thus Eq. 15 can be written in differential form during the process of consolidation, as follows:

$$\delta\bar{\Theta} = \delta u \qquad (16)$$

From Eq. 13 and 16,

$$\delta V_v = -m\,\delta u \qquad (17)$$

The negative sign in Eq. 17 is simply to denote that the volume is decreasing with consolidation.

By substituting Eq. 17 in Eq. 11, the differential equation of consolidation can be written:

$$\nabla\cdot(k\,\nabla u) + Q\gamma_w = m\,\gamma_w \frac{\partial u}{\partial t} \qquad (18)$$

Eq. 18 is the differential equation for a variable permeability and time-dependent loading. Expansion of this equation gives

$$\nabla k\cdot\nabla u + k\,\nabla^2 u + Q\,\gamma_w = m\,\gamma_w \frac{\partial u}{\partial t} \qquad (19)$$

Under the conditions made to this point, Eq. 19 does not sufficiently specify the problem because there are two dependent variables, k and u.

It has recently been proposed by Schmid (*4*) that the coefficient of permeability is related to the porosity by

$$\bar{k} = \beta\,(n - n_0) \qquad (20)$$

This relation is shown in Figure 2.

A considerable body of experimental evidence supports this expression. Schmid's work dealt with the usual concept of permeability as being constant over the length of the flow path. The average coefficient of permeability, $\bar{k}$,

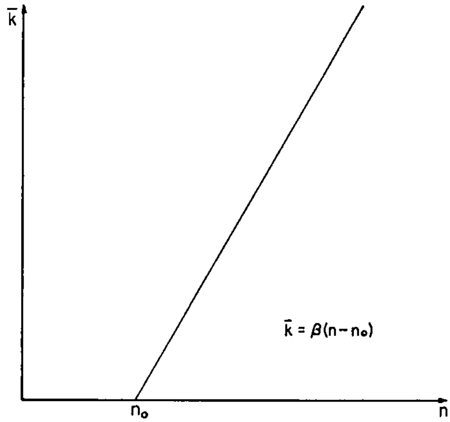

Figure 2. Relationship between coefficient of permeability and porosity.

and the point permeability, k, expressed in this study are related in the following way:

$$\bar{k} = \frac{1}{V}\int_v k\,dV \qquad (21)$$

Assuming the general validity of Eq. 20, as applied to a point function, it then can be used to determine the relation between k and u. Since the phenomenon under investigation is one in which the permeability changes from one finite value to a lower finite value, Eq. 20 must be slightly modified, as follows:

$$k = k_0 - \nu\,\delta V_v \qquad (22)$$

In Eq. 22 the k_0 term is the point function for the permeability at the start of the consolidation process.

Because the relationship between the change in volume of voids and the excess pore pressure is linear, it is logical to extend Eq. 22 so that a relationship is developed between the permeability, k, and the excess pore pressure, u. This relationship is shown in Figure 3, and formulated as follows:

$$k = a\,u + k_f \qquad (23)$$

where k_f is the coefficient of permeability at the end of consolidation.

In terms of the initial values of excess pore pressure and coefficient of permeability, the relationship shown in Figure 3 can be postulated as follows:

$$k = k_0 + a\,(u - u_0) \qquad (24)$$

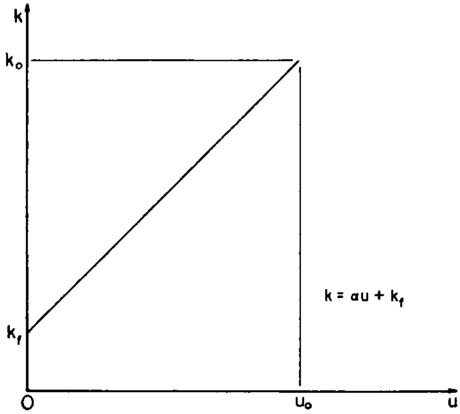

Figure 3. Relationship between coefficient of permeability and excess pore pressure.

In Eqs. 23 and 24, a is defined as the "modulus of permeability variation," which in Eq. 23 must be considered as related to the imposed pressure. Although it is beyond the scope of this study to postulate this relation, the fact of its existence, and the order of dependency, confirms previous experimental work by Taylor (*8*).

The modulus of permeability variation can be related to the modulus of volume change through Eqs. 22 and 23, by use of Eq. 17. Thus,

$$a = \nu m \tag{25}$$

It is the opinion of this author, that ν and m are related by the level of imposed pressure, the soil properties, and possibly the state of structure of the soil. In the absence of any theoretical and/or experimental evidence relating these coefficients, they will be treated as separate entities.

At this point, a differential equation governing the general theory of consolidation, with time-dependent loading and varying permeability, can be established. This differential equation can be written in terms of the excess pore pressure, u, or the permeability, k, as follows:

$$a \nabla u^2 + \nabla u \cdot \nabla k_f + a\, u \nabla^2 u + k_f \nabla^2 u + Q \gamma_w = m \gamma_w \frac{\partial u}{\partial t} \tag{26a}$$

$$\frac{1}{a}(\nabla k)^2 - \frac{1}{a}(\nabla k \cdot \nabla k_f) + \frac{1}{a} k \nabla^2 k - \frac{1}{a} k \nabla k_f + Q \gamma_w = (m \gamma_w / a) \frac{\partial k}{\partial t} \tag{26b}$$

$$\nabla k_0 \cdot \nabla u - a(\nabla u_0 \cdot \nabla u) + a(\nabla u)^2 + a\, u \nabla^2 u + (k_0 + a\, u_0)\nabla^2 u + Q \gamma_w = m \gamma_w \frac{\partial u}{\partial t} \tag{26c}$$

The three forms of the governing differential equation are presented to show the flexibility of representation of the phenomena, and to set up comparisons for later studies in which approximation techniques will be used.

The differential equation (Eq. 26) is complex in the fact that it is non-linear and solutions to non-linear partial differential equations are not at all well established. In fact, only a few particular types of the foregoing equations have been solved, and these solutions have been in terms of complex, untabulated functions. Because of the complexities involved no attempt has been made in this study to develop an exact solution to Eq. 26. There are however, procedures by which this differential equation may be solved without resorting to analytically exact methods. The first, and most obvious method is to solve a given problem numerically, by either analogue or digital computer.

Digital computation techniques for the linear diffusion equation are well defined (*9*). They require that in order to achieve a desired accuracy the time-space mesh must be carefully controlled. Although convergence theorems for non-linear problems can be formulated, the actual convergence in all probability will depend on the magnitude of the constants. Until more information is available concerning the magnitudes of the parameters involved, it was deemed inadvisable to undertake an extensive computational effort.

The use of analogue computers to solve transient problems of the diffusion type requires highly specialized equipment of the circuit analogue type. This equipment is not readily available, and even so would require extensive modification to handle non-linear problems. The necessity of extensive computational efforts is abrogated in large part by the fact that techniques are available to linearize the differential equation (Eq. 26) and thus an approximate analytical solution can be obtained. The advantages of approximate techniques and solutions far outweigh the necessity for computations on the exact equation.

Approximation Procedures

The general formulation of the consolidation problem, in which the permeability is postulated to be a linear function of the excess pore pressure, leads to non-linear terms in the differential equation. To solve this equation, two different approximation procedures are proposed.

The first procedure sets the condition that the coefficient of permeability is initially constant throughout the soil mass and that, over a finite increment of time, it is also constant. The space constancy condition is akin to considering that the concern is with the average permeability over a finite time increment. For this condition the differential equation is reduced as follows:

$$k_i \nabla^2 u_i + Q \gamma_w = m \gamma_w \frac{\partial u_i}{\partial t_i} \qquad i = 1,2,3 \ldots \tag{27}$$

Eq. 27 is valid over the ith finite time increment and is solved by considering that the solution for the $(i-1)$st increment is the initial condition to the ith increment.

The second procedure was developed by Charney (*10*) for heat conduction problems and is applicable here in all details. This development extends this approximation to cases of time-dependent loading.

Basically, the problem can be reformulated by Darcy's law and the continuity condition in coupled form, as follows:

$$\gamma_w \vec{v} = k(u)\,[\nabla u] \tag{28a}$$

$$\nabla \cdot \vec{v} + Q = m \frac{\partial u}{\partial t} \tag{28b}$$

Since u is a function of k, by a chain rule differentiation:

$$\nabla u = \frac{du}{dk} \nabla k \tag{29}$$

and

$$\frac{\partial u}{\partial t} = \frac{du}{dk}\frac{\partial k}{\partial t} \tag{30}$$

Thus the Eqs. 28a and 28b take the form

$$\gamma_w \vec{v} = k \frac{du}{dk} \nabla k \tag{31a}$$

$$\nabla \cdot \vec{v} + Q = m \frac{du}{dk}\frac{\partial k}{\partial t} \tag{31b}$$

Then the governing differential equation becomes:

$$\nabla \cdot \left[k \frac{du}{dk} \nabla k \right] + Q \gamma_w = m \gamma_w \frac{du}{dk}\frac{\partial k}{\partial t} \tag{32}$$

In order to reduce this equation to a linear form, the following conditions will be formulated:

$$k \frac{du}{dk} = \text{Constant} = \rho \tag{33a}$$

$$\frac{du}{dk} = \frac{1}{a} \tag{33b}$$

Eq. 33b is completely compatible with the permeability-pore pressure relationship postulated previously in Eq. 23.

It is obvious that Eqs. 33a and 33b are rigorously incompatible with each other, inasmuch as Eq. 33a is an exponential function and Eq. 33b is a linear function. It is in the handling of these conditions that the approximation is made. For purposes of approximation, it will be postulated that the exponential in a $k-u$ graph approximates the linear relation. This condition is shown in Figure 4.

The solution to Eq. 33a over a range of k and u is:

$$\rho = \frac{u_0 - u}{\ln(k_0/k)} \tag{34}$$

Combining Eqs. 32, 33, and 34, the general problem can be formulated in a new linear differential equation on k, as follows:

$$B \nabla^2 k + \frac{a}{m} Q = \frac{\partial k}{\partial t} \tag{35a}$$

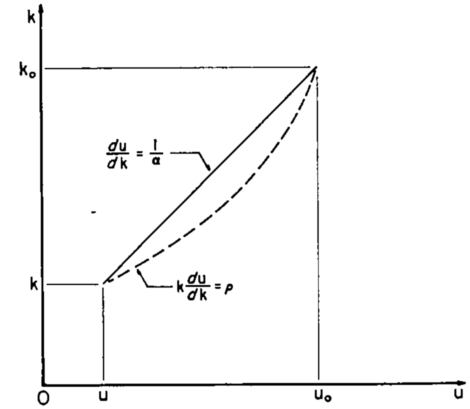

Figure 4. Exponential approximation for coefficient of permeability-excess pore pressure relationship.

in which

$$B = \frac{a\,p}{m\,\gamma_w} \tag{35b}$$

The B term has the same dimensions as the coefficient of consolidation and is defined as the "coefficient of consolidation permeability."

Several procedures are possible for the solution to problems involving varying permeability, using the approximation in Eq. 35. The first suggested method is to solve for k over the entire time range and then to convert k to u, either by the linear approximation or by the exponential approximation. Another procedure, which is somewhat more tedious, is to fit the approximations over small finite time intervals. Final adoption of a proper approximation procedure probably must await experimental evidence as to the goodness of fit between the linear and exponential curves shown in Figure 4.

ONE-DIMENSIONAL CONSOLIDATION

To illustrate the solution to the theory of consolidation with varying permeability and subject to time-dependent loading, the classic case of a doubly-drained layer of clay-soil will be considered (Fig. 5). In this problem the clay layer is of finite thickness, and infinite in width and breadth, making up a thick plate. The loading on the surface is infinite in extent, but can be time-dependent. In all cases the initial coefficient of permeability and the initially imposed excess pore pressure will be postulated to be uniform throughout the clay-soil.

Under these conditions the governing differential equation reduces to

$$\left[C_0 + \frac{a\,u_0}{m\,\gamma_w}\right]\frac{\partial^2 u}{\partial z^2} - \frac{a}{m\,\gamma_w}u\frac{\partial^2 u}{\partial z^2} - \frac{a}{m\,\gamma_w}\left[\frac{\partial u}{\partial z}\right]^2 + R = \frac{\partial u}{\partial t} \tag{36}$$

in which R is the rate of change of imposed excess pore pressure, u', and C_0 is the coefficient of consolidation at the start of the process of consolidation.

The difficulty in the solution to Eq. 36 lies in its non-linear terms. As stated previously, no attempt was made to solve these types of equations exactly. Solutions using approximation techniques are developed in subsequent sections.

The solution to the general problem will be broken down into the solution of several individual problems, as follows:

1. Constant permeability and general time-dependent loading.

2. Constant permeability and linear loading.

3. Constant permeability and construction loading.

4. Constant permeability and harmonic loading.

5. Variable permeability.

Constant Permeability, Time-Dependent Loading

The general time-dependent loading problem for a condition of one-dimen-

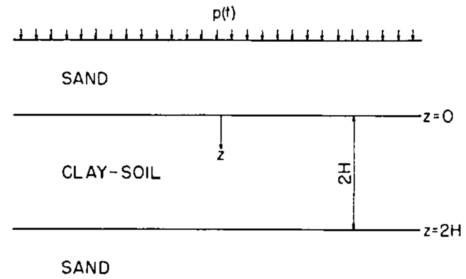

Figure 5. Double-drained clay layer, one dimension.

sional consolidation where the permeability is held constant, reduces the differential equation (Eq. 36) as follows:

$$C\frac{\partial^2 u}{\partial z^2}+R=\frac{\partial u}{\partial t} \qquad (37)$$

The conditions on the doubly-drained layer enabling a solution are as follows:

1. The surfaces joining the sand and the clay are free-draining and therefore will be free of excess pore water pressure, u, for all time t, or

$$u(0,t)=0 \qquad 0\leqq t\leqq\infty \qquad (38a)$$

$$u(2H,t)=0 \qquad 0\leqq t\leqq\infty \qquad (38b)$$

2. The general initial condition will be that the vertical stress imposed by the load at the start of consolidation will be some function of space within the clay layer.

$$u(z,0)=\sigma(z) \qquad 0\leqq z\leqq 2H \qquad (39)$$

3. The rate of loading, R, will be interpreted as the rate of imposition of excess pore pressure and will, in this general case, be considered as a function of time, t and space, z.

There are several formal procedures for solving this boundary value problem. All these procedures have roots in the classical Sturm-Liouville solutions to the differential equations of physics. The method chosen for use here was that of integral transforms *(11)*. This method was chosen above other, possibly better known procedures because it enabled the handling of boundary value problems in which forcing functions, R, are present, in a highly formalized manner. By this method, once the transforms are set up, the solution is directly obtainable.

By use, in this case, of the Fourier sine transform, the following solution was developed:

$$u(z,t)=\frac{1}{H}\sum_{n=1}^{\infty}\left[\int_0^{2H}\sigma(z)\,\sin\frac{n\pi}{2H}z\,dz\right]\sin\frac{n\pi}{2H}z\,e^{Mt}+\frac{1}{H}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2H}\,z\left\{\int_0^t\left[\int_0^{2H}R(z,t)\,\sin\frac{n\pi}{2H}z\,dz\right]e^{M(t-\tau)}d\tau\right\} \qquad (40a)$$

in which

$$M=-C\,\frac{n^2\pi^2}{4H^2} \qquad (40b)$$

Thus, by the evaluation of three relatively simple integrals, the excess pore pressure can be determined.

It may be of some interest to note that this solution is in actuality the sum of the usual constant load consolidation problem and a time-dependent loading problem where the initial condition does not enter. Thus, many problems may be developed by judicious combinations of existing solutions.

Constant Permeability, Linear Loading

The linear loading problem is graphically represented in Figure 6. It will be presumed that the surface loading, p, is imposed at a constant rate, R. Thus, for a specified load and time the rate is

$$R=p_0/t_0 \qquad (41)$$

Furthermore, a reasonable condition on the physics of the problem will be that the rate of imposition of pore pressure is the same as the rate of imposition of surface loading. Thus, Eq. 41 can also be written:

$$R=u_0/t_0 \qquad (42)$$

in which u_0 is the imposed excess pore pressure at time t_0.

A solution to the consolidation problem with linear loading can be developed

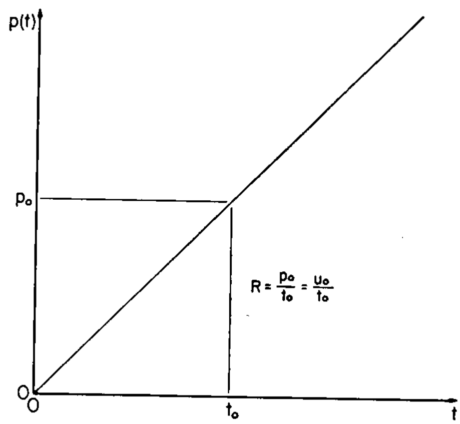

Figure 6. Linear loading.

directly by substitution in Eq. 40. Because there is no initial excess pore pressure,

$$\sigma = 0 \tag{43}$$

The solution for the excess pore pressure at any point in the soil mass is:

$$u(z,T) = \frac{16u_0}{T_0\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2H} z[1 - e^{-(n^2\pi^2/4)T}] \tag{44}$$

The average excess pore pressure is defined as:

$$\bar{u}(T) = \frac{1}{2H} \int_0^{2H} u(z,T)\,dz \tag{45}$$

from which

$$\bar{u}(T) = \frac{32\bar{u}_0}{T_0\,\pi^4} \sum_{n=1,3,5}^{\infty} \frac{1}{n^4} [1 - e^{-(n^2\pi^2/4)T}] \tag{46}$$

In Eqs. 44, 45, and 46 the time is established in terms of the usual dimensionless time factor, T:

$$T = \frac{C}{H^2} t \tag{47}$$

A computation was performed on Eq. 46 so that useful curves could be developed from the analytical results. Figure 7 is a graph of the dimensionless representation of excess pore pressure, $\bar{u}/\bar{u}_0$, times the reference time factor, T_0, versus the time factor, T. This is the basic curve, and from it all other subsequent computations can be made for the linear case.

There are many instances where the soil conditions are such that structural designs and soil treatment are inadequate measures for the control of settlement. In these cases it is sometimes feasible to preload the soil and, after a designated period of time when the objectionable portion of the settlement has occurred, the preload is removed and the structure built without serious settlement prob-

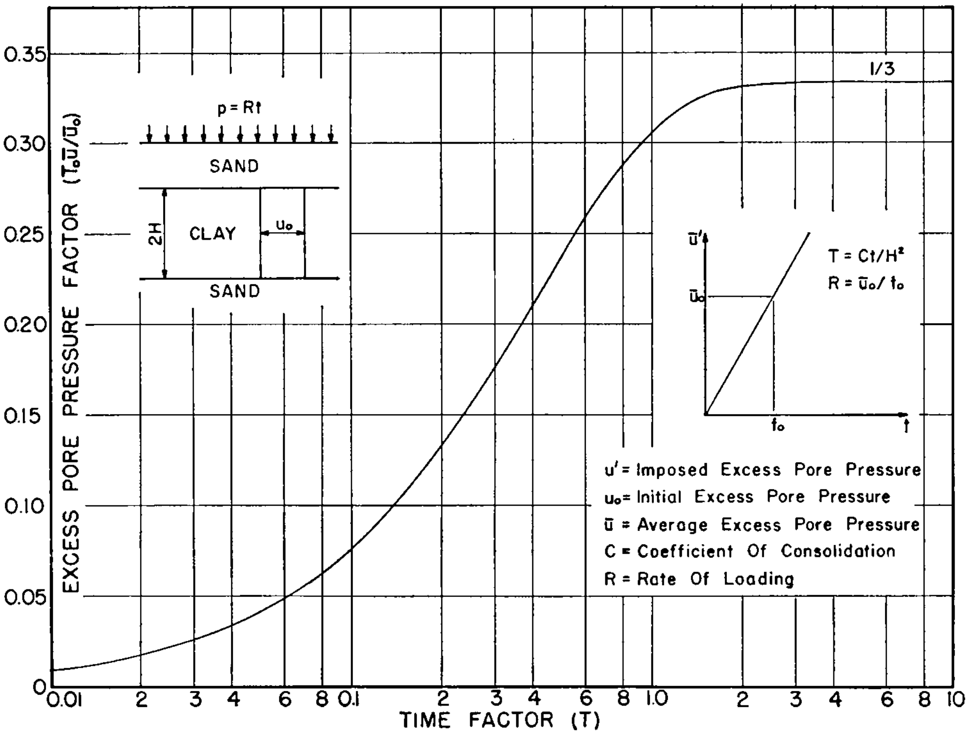

Figure 7. Linear loading consolidation.

lems. Another procedure that has been considered is to use the construction period as a means of controlling settlement. If the construction period is thought of as a linear loading period ending at time t_0, the amount of excess pore pressure, and thus the percentage of settlement dissipated, at the end of a linear construction period can be computed. Such a computation is presented in Figure 8. This curve is presented in terms of the dimensionless end time factor, T_0, which is nothing more than the time factor at the end of the period of load build-up.

An example taken from actual laboratory data considers a clay-soil layer 40 ft thick. The clay has a coefficient of consolidation of 1.5 sq cm/min. For this value, the duration of construction, t_0, will be related to the end time factor, T_0, as follows:

$$t_0 = 172\, T_0 \text{ Days}$$

By entering Figure 8, a table of percentage of settlement dissipated can be constructed in terms of the time of construction. On the basis of Table 1, in this example the soil engineer is now equipped to analytically determine what the most

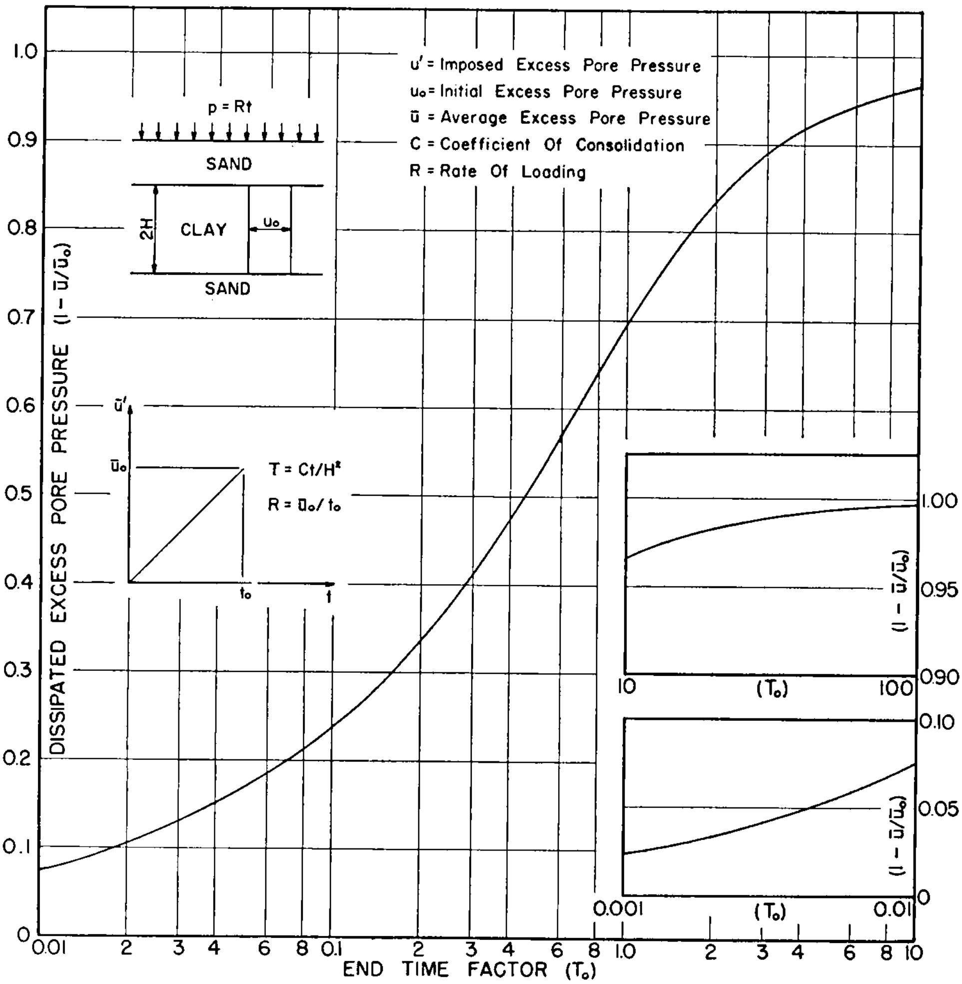

Figure 8. Excess pore pressure dissipated at end of linear loading period.

TABLE 1

EXAMPLE OF SETTLEMENT DISSIPATED DURING A LINEAR CONSTRUCTION PERIOD

($H = 20$ ft; $C = 1.5$ sq cm/min)

% Dissipated Settlement	End Time Factor, T_0	Time, t_0 (days)
5	0.00435	0.75
10	0.018	3
20	0.071	12
30	0.16	28
40	0.28	49
50	0.45	78
60	0.68	117
70	1.03	177
80	1.65	284
90	3.25	559
95	6.70	1,153

desirable construction period would be to maintain settlements within non-objectionable values. Extending this example, suppose the nature of the structure was such that the post-construction settlements had to be 30 percent of the total magnitude. To accomplish this objective a construction period of 6 months would be required. On this basis it would now be possible to determine the relative economics of preloading, artificially lengthened construction, and other means of controlling objectionable settlement.

Constant Permeability, Construction Loading

Using the results of the preceding sections, the full excess pore pressure history can be analyzed at any point in time and for any length of construction.

A typical load-time diagram for a construction loading is presented in Figure 9. In this problem, the application of construction loading is considered to be linear from the start of construction to

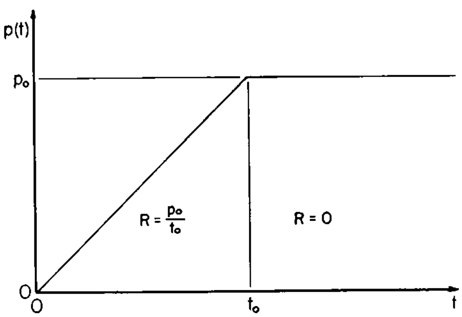

Figure 9. Construction loading.

the end, at which time a load of p_0 is imposed on the surface. The imposed excess pore pressure, u' increases at the same rate, R, as the surface loading and culminates in a total excess pore pressure of u_0. Furthermore, the rate of imposition of pore pressure is constant throughout the soil mass.

The solution for the excess pore pressure during construction is identical to the previously described linear loading problem as formulated in Eqs. 44 and 46. These equations must be considered to be valid only when $T \leqq T_0$.

For periods beyond the construction period, the pore pressure is governed by the usual Terzaghi differential equation. The initial condition applied to this differential equation is the pore pressure at the end of construction. This solution, which is valid for all times beyond the end of construction, is:

$$u(z,T) = \frac{16u_0}{T_0\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2H} z \left[1 - e^{-(n^2\pi^2/4)T_0}\right] e^{-(n^2\pi^2/4)(T-T_0)} \quad (48)$$

$$\bar{u}(T) = \frac{32\bar{u}_0}{T_0\pi^4} \sum_{n=1,3,5}^{\infty} \frac{1}{n^4} \left[1 - e^{-(n^2\pi^2/4)T_0}\right] e^{-(n^2\pi^2/4)(T-T_0)} \quad (49)$$

Sets of curves (Figs. 10a and 10b) for determining the average excess pore pressure during and after construction were computed on the basis of Eqs. 46 and 49.

A specialized solution for time-dependent loading, applicable only in the one-dimensional case, was developed by Terzaghi and Frohlich (*2*). Using their solution, Wilson and Grace (*12*) have presented a computation for the linear construction loading. The results in Figures 10a and 10b are essentially the same computation, although performed independently, and in a form that is considered to be more convenient. In addition, the computations presented in this paper were made with modern electronic computing equipment, and therefore have a higher magnitude of accuracy. These curves can be used to determine the theoretical consolidation curve. The actual

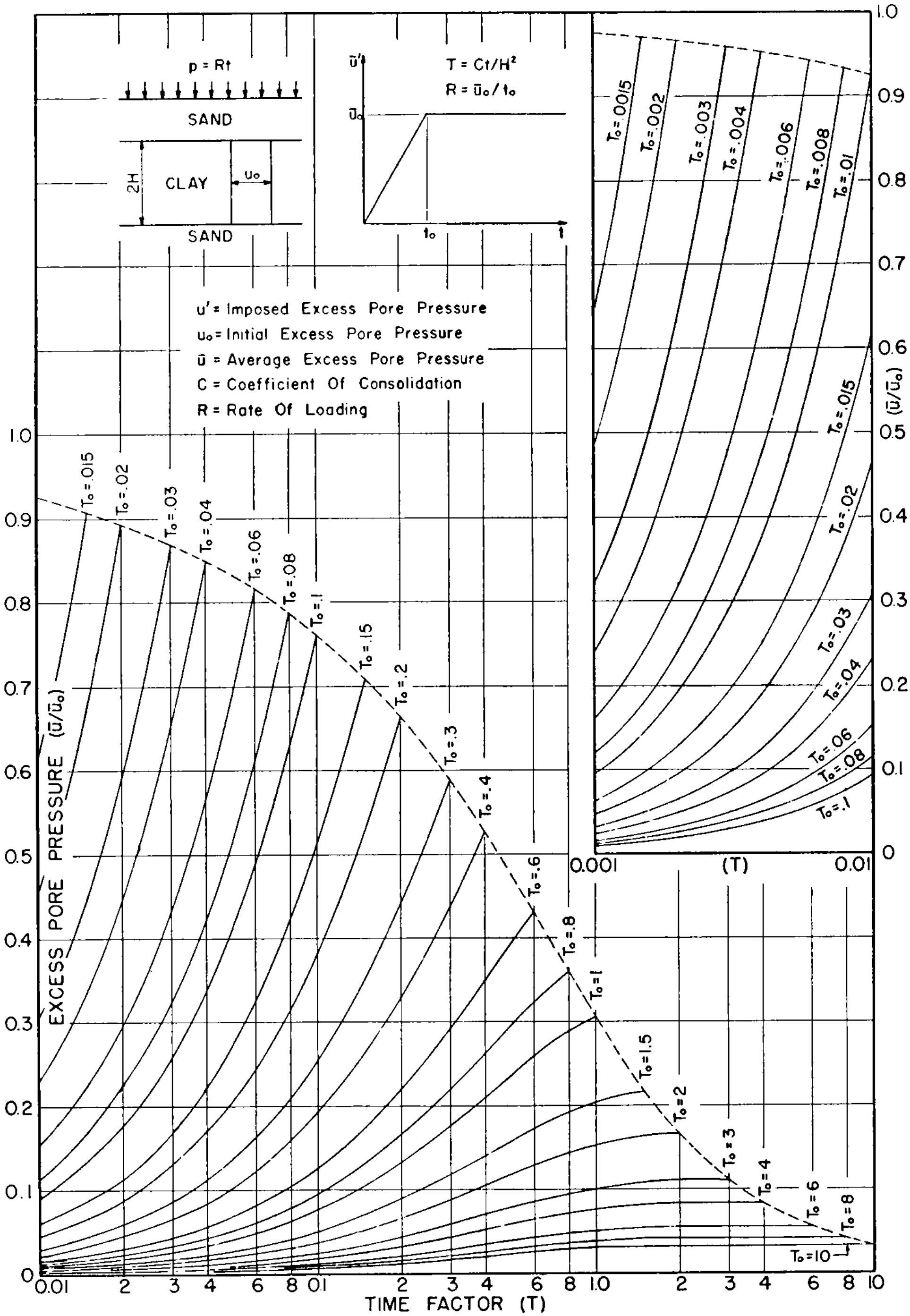

Figure 10a. Consolidation during construction.

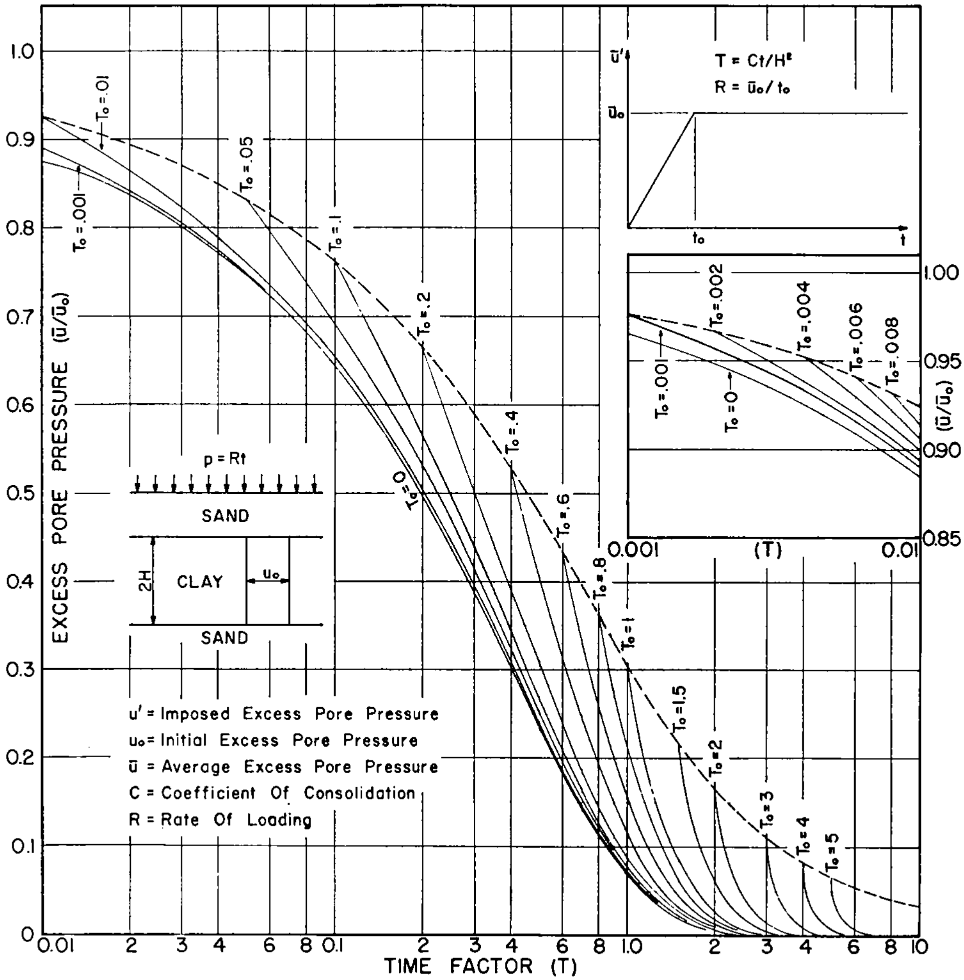

Figure 10b. Consolidation subsequent to construction.

construction time can be estimated. By laboratory tests the coefficient of consolidation can be determined, and with a knowledge of the soil profile the end time factor, T_0, is computable. By entering Figs. 10a and 10b, the proper consolidation curve is then selected. Beyond this point the analysis follows the usual practice for time-settlement predictions.

The determination of the coefficient of consolidation and the laboratory test procedure that is necessary is not clearly defined, and will not be until an exhaustive experimental analysis has been conducted with this theory as a base of operation.

Experimental work by Taylor (*8*) for time-dependent loading has indicated that the coefficient of consolidation is dependent on the loading rate as well as the final load level. In the absence of more detailed experimental work, it is tentatively suggested that laboratory tests for actual field problems be conducted at the stress levels and stress rates expected in the field.

Constant Permeability, Harmonic Loading

The physical nature of the consolidation process is one that has direct analogy

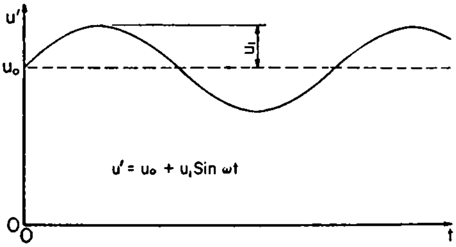

Figure 11. Slow harmonic loading.

with the theories of rheology. Attempts have been made to analyze the problem of consolidation in terms of a visco-elastic mechanical model (*13*). On this basis the theory of consolidation must be restricted to load conditions where the time-rate of loading is slow. The theory will not treat impact problems or transient load problems of high frequency, except as steady-state loading. There is, however, an intermediate class of problems in which harmonic loads are imposed on the soil mass with a low frequency. These problems are easily handled by the theory presented.

As a practical example, consider the case of an industrial plant with service by heavily loaded railroad cars. The loading is imposed for a finite time period and then removed for a finite time period. In many instances, such loading can be represented as a slow harmonic loading, as shown in Figure 11. The loading function is:

$$u'(t) = u_0 + u_1 \sin \omega t \qquad (50)$$

The rate of loading is, therefore:

$$R(t) = u_1\omega \cos \omega t \qquad (51)$$

in which u_1 is the amplitude of imposed harmonic excess pore pressure, and ω is the frequency of loading.

The solution to this particular problem uses the full Eq. 40 with

$$\sigma(z) = u_0 \qquad (52)$$

The general solution in the one-dimensional case is:

$$u(z,t,T) = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\,\pi}{2H}\, z\, e^{-(n^2\pi^2/4)T}$$

$$+ (16C\, H^2 \pi\omega u_1) \sum_{n=1,3,5}^{\infty} \frac{n}{\beta_n}\left[\cos \omega t \sin \frac{n\,\pi}{2H} z - e^{-(n^2\pi^2/4)T}\right]$$

$$+ \frac{64H^4 u_1}{\pi}\omega^2 \sin \omega t \sum_{n=1,3,5}^{\infty} \frac{1}{n\beta_n} \sin \frac{n\,\pi}{2H} z \qquad (53)$$

$$\bar{u}(t,T) = \frac{8\bar{u}_0}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} e^{-(n^2\pi^2/4)T}$$

$$+ 32C\, H^2\, \bar{u}_1\omega \cos \omega t \sum_{n=1,3,5}^{\infty} \frac{1}{\beta_n}$$

$$+ \frac{128H^4}{\pi^2} \bar{u}_1\omega^2 \sin \omega t \sum_{n=1,3,5}^{\infty} \frac{1}{n^2\beta_n}$$

$$- 16C\, H^2\, \pi\omega\bar{u}_1 \sum_{n=1,3,5}^{\infty} \frac{n}{\beta_n} e^{-(n^2\pi^2/4)T} \qquad (54)$$

$$\beta_n = C^2 n^4 \pi^4 + 16\omega^2 H^4 \qquad (55)$$

Eqs. 53, 54, and 55 are in an unreduced form, and as such may be considered to be overly complex with regard to the solution of an actual problem. Another approach to this problem is to look upon the harmonic load problem as a loading in terms of a step function, as indicated in Figure 12. The solution to the situation of step loading need not use the time-dependent loading solutions, but can be treated as a succession of steady loads and incomplete consolidation. The general class and solution to these problems is covered in the sections on varying permeability.

Variable Permeability

The first approximation technique that was developed was the incremental time approximation. In this it was assumed that the permeability of the stratum remained constant over a finite time increment, but that the permeability varied from one time increment to the next. Because the modulus of volume change is set to remain constant, the variations in the coefficient of permeability are directly proportional to the variations in the co-

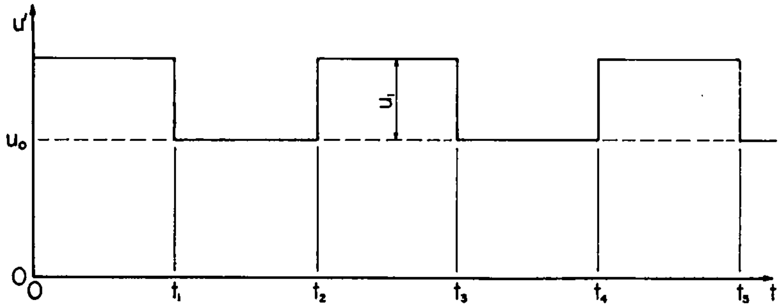

Figure 12. Step loading.

efficient of consolidation. Thus, the basic time variation is as shown in Figure 13.

Essentially, the solution for the constant loading case using this constant incremental approximation is solved for each increment, using the excess pore pressure at the end of the previous time increment as the initial condition. Thus, for the ith time increment,

$$C_i \frac{\partial^2 u_i}{\partial z^2} = \frac{\partial u_i}{\partial \bar{t}} \qquad (56a)$$

$$u_i(0,\bar{t}) = 0 \qquad 0 \leqq \bar{t} \leqq \infty \qquad (56b)$$

$$u_i(2H,\bar{t}) = 0 \qquad 0 \leqq \bar{t} \leqq \infty \qquad (56c)$$

$$u_i(z,0) = u_{i-1}(z,t_i) \qquad 0 \leqq z \leqq 2H \qquad (56d)$$

in which

$$\bar{t} = t - t_i \qquad (56e)$$

Thus the solution for the ith range of time ($\bar{t}$) is valid only in this increment and has as its initial condition the excess pore pressure at the end of the $(i-1)$st increment.

The actual solution to this problem for a finite number of time increments can be developed by successive solutions to the boundary value problem in Eq. 56. This solution is:

$$u(z,T') = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2H} z \, e^{-(n^2\pi^2/4)T'} \qquad (57a)$$

$$\bar{u}(T') = \frac{8\bar{u}_0}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} e^{-(n^2\pi^2/4)T'} \qquad (57b)$$

$$T' = \frac{1}{H^2} [C_1 t_1 + C_2(t_2 - t_1) + C_3(t_3 - t_2) + \ldots + C_m(t - t_m)] \qquad (57c)$$

The usage of this approximation in computation is similar to the usual curve fitting procedures developed by Casagrande (*14*). The only difference comes about in the fact that the laboratory curve is fitted in increments, whereas previous procedures had the curves fitted once for the entire load range. The test procedure is the usual one with time compression readings being taken at pre-

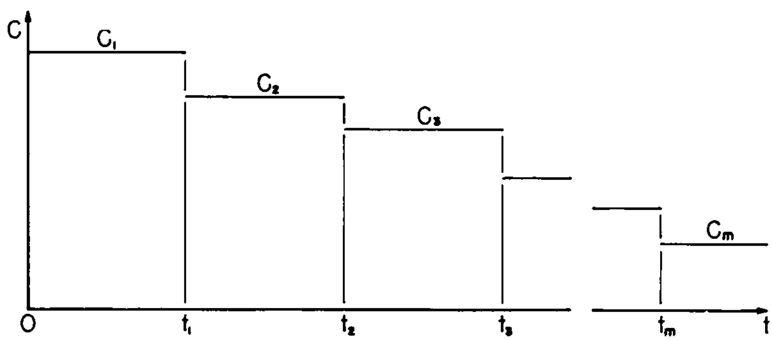

Figure 13. Segmental time approximation for coefficient of consolidation.

determined finite increments. At the end of a given increment of time the laboratory and theoretical curve can be fitted on the basis of the theoretical values of $\bar{u}$ from Eq. 57*b* and the time factor, T', from Eq. 57*c*. When the fit is carried out from $t = 0$ on, successive values of the bracketed term in Eq. 57*c* can be determined. For the first time increment, t_1, the value of C_1 is determined. For the second time increment, the value C_2 can be determined, inasmuch as C_1 is known. This procedure can be carried on successively until all time increments are completed and thus all the individual coefficients of consolidation are known.

This approximation procedure is quite versatile in that the degree of accuracy used is a matter of decision on the part of the investigator. If a highly accurate fit is desired a large number of time segments can be used. If, on the other hand, a lower degree of precision is required the number of curve fits can be arbitrarily reduced.

The second approximation developed for the consideration of variable permeability was one in which the permeability-pore pressure relation, originally postulated to be linear, was approximated in an exponential fashion, as shown in Figure 4.

The solution to this problem can be handled in several ways. The first and possibly the crudest of this type of approximation is to consider that the approximation is valid over the entire consolidation range. In establishing this condition, one can obtain a specific solution for the pore pressure variations over the entire range of time, and also an expression for the error of approximation.

For the first solution the case of time-independent loading is considered. This problem is governed by the following differential equation:

$$B\frac{\partial^2 k}{\partial z^2} = \frac{\partial k}{\partial t} \qquad (58)$$

The boundary value problem is defined by the pore pressure boundary and initial conditions expressed in terms of the coefficient of permeability:

$$k(0,t) = k_f \qquad 0 \leqq t \leqq \infty \qquad (59a)$$

$$k(2H,t) = k_f \qquad 0 \leqq t \leqq \infty \qquad (59b)$$

$$k(z,0) = \alpha\, u_0 + k_f \qquad 0 \leqq z \leqq 2H \qquad (59c)$$

The solution to the boundary value problem is (*15*):

$$k(z,t) = k_f + \frac{4\alpha u_0}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2H}z\, e^{-(n^2\pi^2/4H^2)\,B\,t} \qquad (60)$$

Converting the solution, Eq. 60, back to excess pore pressure terminology,

$$u(z,t) = \frac{4u_0}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2H}z\, e^{-(n^2\pi^2/4H^2)\,B\,t} \qquad (61)$$

Eq. 61 is identical to the usual solution for the consolidation of fine-grained soils, with a single change in notation. In this solution the coefficient of consolidation, C, is replaced by the coefficient of consolidation permeability, B.

It can readily be determined that this approximation, for all cases of time-dependent and time-independent loading, is identical in form to the constant permeability solution with a single exception. This exception is in the explicit definition of the time factor, T, considering this approximation technique. The time factor, T, is replaced by a new time factor, V, which considers the permeability variations. This new time factor, defined as the "time factor for variable permeability," is

$$V = \frac{B}{H^2}\, t \qquad (62)$$

Thus, by replacing T and T_0 with V and V_0 in Eqs. 44, 53, and 54, the condition of variable permeability can be introduced in a simple manner.

The expression for the coefficient of consolidation permeability, B, can be determined from the definition of the constants α in Eq. 23 and ρ in Eq. 34. Fitting the approximation in Figure 4 at the end points k_0 and k_f, which correspond to the initial and final excess pore pressures, u_0

and 0, respectively, the value of B becomes

$$B = \frac{k_0 - k_f}{\eta m \gamma_w} \tag{63a}$$

in which

$$\eta = \ln \frac{k_0}{k_f} \tag{63b}$$

It may be of interest to note that if k_0 approaches k_f the value of B approaches the usual coefficient of consolidation, C; that is,

$$\lim_{k_0 \to k_f} B = \frac{k_f}{m \gamma_w} = C \tag{64}$$

Examination of Eq. 63 gives a clearer picture of the true nature of the coefficient of consolidation. Many investigators in using this coefficient have used the average value of the permeability and have claimed a good measure of success for this method in determining the settlement. Others have disputed this use of averages, and have expressed the opinion that the theory of consolidation is somewhat limited in validity. This limitation has been particularly stated for soils with highly flocculent structures where the differences in the permeability before and after loading are large. It is not mere coincidence that the values of B in Eq. 63, and of C in the usual theory, diverge as the difference between k_0 and k_f become greater.

A measure of the difference between the usual concept of mean permeability over the consolidation period and the approximate linear theory can be represented by the ratio of the differences between the constants C and B as a measure of the differences in the range of the coefficients of permeability.

If it is assumed that

$$E = \frac{C - B}{B} \tag{65a}$$

then

$$E = \frac{\mu + 1}{2(\mu - 1)} \ln \mu - 1 \tag{65b}$$

in which

$$\mu = k_0 / k_f \tag{65c}$$

In a plot of E against μ (Fig. 14), it should be noted that if the initial and final permeabilities differ by a factor of 10 the difference between the two methods of evaluation will be on the order of 41 percent. A difference of 135 percent occurs when the permeabilities differ by a factor of 100.

In a discussion of errors, the absolute error between the linear postulation and the exponential approximation should be determined. This error, ϵ, can be computed by determining the difference between the linear relation and the exponential approximation for the relationship between permeability and excess pore pressure. The maximum error as a fraction of the initial excess pore pressure is

$$\frac{\epsilon_{max}}{\bar{u}_0} = 1 - \frac{1}{\eta} \left[\ln \frac{\mu \eta}{\mu - 1} + \frac{\eta}{\mu - 1} - 1 \right] \tag{66}$$

If the error defined by Eq. 66 is intolerable, the precision of the approximation can be increased by segmentally fitting the linear curve with exponentials, as shown in Figure 15.

The solution for the constant load problem follows the segmentally constant permeability solution identically and is:

$$u(z, V') = \frac{4u_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2H} z \, e^{-(n^2\pi^2/4)V'} \tag{67a}$$

$$\bar{u}(V') = \frac{8\bar{u}_0}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} e^{-(n^2\pi^2/4)V'} \tag{67b}$$

$$V' = \frac{1}{H^2}[B_1 t_1 + B_2(t_2 - t_1) + B_3(t_3 - t_2) + \dots + B_m(t - t_m)] \tag{67c}$$

$$B_i = \frac{1}{m\gamma_w}[(k_{i-1} - k_i)/\ln(k_{i-1}/k_i)] \tag{67d}$$

In this approximate theory the basic solution is for the point coefficient of permeability, in terms of the terminal coefficients of permeability, k_0 and k_f. For the theory to be consistent the terminal coefficients must be point values, and thus will differ at different points in the soil mass. In integrating the point values of the excess pore pressure to get the average values, these point functions

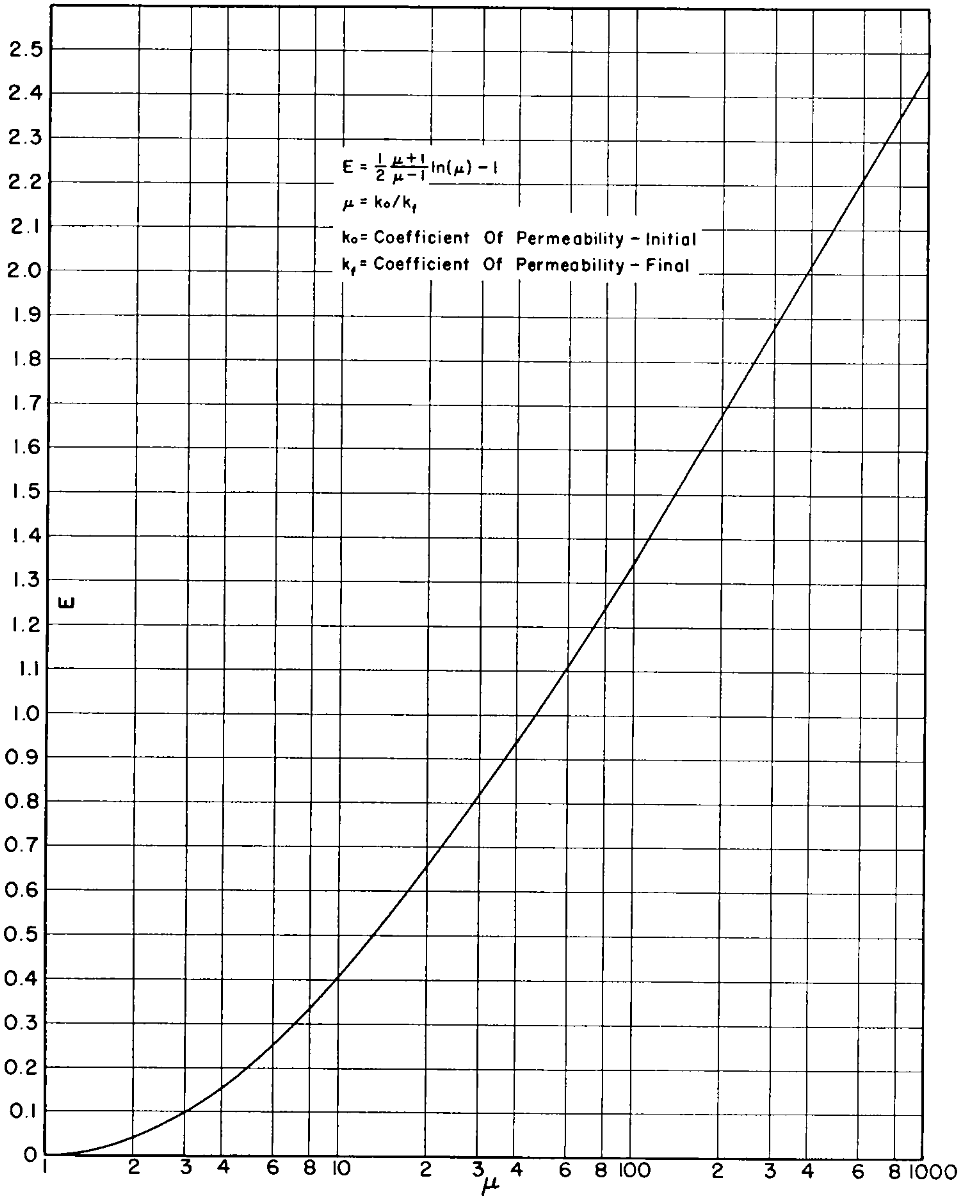

Figure 14. Error between coefficient of consolidation and coefficient of consolidation permeability.

of terminal permeability were carried without change. This could only have been done if these terminal values were constant throughout the soil mass. Thus, in the foregoing theory a condition that always must be kept in mind is that the soil has an initial and final coefficient of permeability which is constant throughout the mass. The same condition must be applied to the terminal points in any segment.

TWO-DIMENSIONAL LOADING

The one-dimensional consolidation problem fits a real situation only where

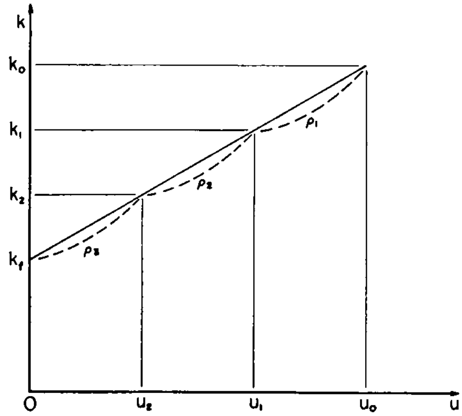

Figure 15. Segmental exponential approximation for coefficient of permeability-excess pore pressure relation.

the compression, and thus the drainage, is in one direction. This in reality requires that the fine-grained soil be infinite in extent in two of the three Cartesian coordinate directions and, furthermore, the load be infinite in extent. Although there are many situations where blanket loading is applied in the field, the vast majority of cases comprises those in which the layer of fine-grained soil is loaded over a finite region.

The first problem of a load over a finite area that is considered is that of a strip load, the general problem considered being a doubly-drained clay layer, as shown in Figure 16.

The solutions to problems of variable permeability in two and three dimensions are essentially solved by the same methods as in one dimension. For this reason, the formal solutions presented in this section are only for constant permeability.

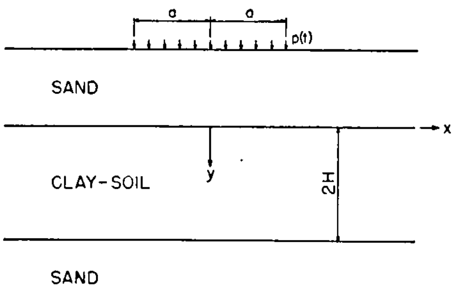

Figure 16. Double drainage clay layer, two dimensions.

The variable load, constant permeability consolidation problem is defined by the following diffusion equation, in which the soil is considered to be bi-dimensionally anisotropic:

$$k_x \frac{\partial^2 u}{\partial x^2} + k_y \frac{\partial^2 u}{\partial y^2} + Q\gamma_w = m\gamma_w \frac{\partial u}{\partial t} \quad (68)$$

Eq. 68 can be converted to isotropic conditions by making the usual scale transformation, in which the horizontal permeability is considered to be greater than the vertical permeability. Thus,

$$z = x\sqrt{k_y/k_x} \quad (69a)$$

$$C_y \left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} \right] + R = \frac{\partial u}{\partial t} \quad (69b)$$

$$C_y = k_y/m\,\gamma_w \quad (69c)$$

The boundary conditions for this problem are as follows:

1. The boundary surfaces between the sand and the clay are free draining and therefore free of excess pore pressure:

$$u(z,0,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq |z| \leqq A \quad (70a)$$

$$u(z,2H,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq |z| \leqq A \quad (70b)$$

2. The motivation for consolidation is the imposed stress. The loading over a finite region of the boundary will dissipate to some small stress at a finite distance from the loaded area. In general terms and in terms of the transformed section, it is presumed that the imposed stresses are sufficiently reduced at a distance $z = A$. Considering the symmetry of the system, the lateral boundary conditions are thus:

$$\frac{\partial u}{\partial z}(0,y,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq y \leqq 2H \quad (71a)$$

$$u(A,y,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq y \leqq 2H \quad (71b)$$

3. The initial conditions will be some function of space:

$$u(z,y,0) = f(z,y) \qquad 0 \leqq y \leqq 2H \qquad 0 \leqq |z| \leqq A \quad (72)$$

4. As considered in the prior problems, the rate of loading, R, is interpreted as the rate of imposition of excess pore pressure. In general the loading rate is a function of space and time.

By attacking the problem in terms of successive Fourier sine and cosine transforms, a general solution can be obtained, as follows:

$$u(z,y,t) = \frac{2}{A\,H}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} F_0 \sin\frac{n\pi}{2H}y \cos\frac{(2m+1)\pi}{2A} z\, e^{-C_v \zeta t} + \frac{2}{A\,H}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \sin\frac{n\pi}{2H}y \cos\frac{(2m+1)\pi}{2A} z \int_0^t \Lambda\, e^{-C_v \zeta(t-\tau)}\, d\tau \quad (73a)$$

in which

$$F_0 = \int_0^A \int_0^{2H} f(z,y) \sin\frac{n\pi}{2H} y \cos\frac{(2m+1)\pi}{2A} z\, dy\, dz \quad (73b)$$

$$\Lambda = \int_0^A \int_0^{2H} R(z,y,t) \sin\frac{n\pi}{2H} y \cos\frac{(2m+1)\pi}{2A} z\, dy\, dz \quad (73c)$$

and

$$\zeta = \frac{(2m+1)^2\pi^2}{4A^2} + \frac{n^2\pi^2}{4H^2} \quad (73d)$$

As before, the solutions to specific problems now can be evaluated by the evaluation of the integrals in Eq. 73.

The solutions will be broken down into two types of problems, those solved being as follows:

1. Constant imposed load.
2. Linear imposed loading.

The stress distribution in the clay layer is a subject of some interest. Although the vertical distribution of stress can be considered to be constant, the lateral distribution must dissipate to 0 at the distance A from the origin of coordinates. This lateral distribution can be postulated by the theory of elasticity. As a first approximation to this theory, two limiting cases are considered:

$$f_1(z,y) = u_0[1 - z/A] \quad (74a)$$

$$f_2(z,y) = u_0[1 - (z/A)^2] \quad (74b)$$

These distributions are shown in Figure 17.

Constant Load

The constant load, two-dimensional consolidation problem can be solved by evaluating the integral and Eq. 73b and using the first term of the series of Eq. 73a. In the evaluation of the integral, using the proper loading function, two time factors are required: T_v is the factor for vertical consolidation, whereas T_x is the time factor for horizontal drainage, such that

$$T_v = \frac{C_v}{H^2} t \quad (75a)$$

$$T_x = \frac{C_v}{A^2} t \quad (75b)$$

The major interest is the time of vertical drainage. In order to carry a single time factor, T_v and T_x are related by a parameter, expressing the ratio of the time factors:

$$\phi = \frac{T_x}{T_v} = \frac{H^2}{A^2} \quad (76)$$

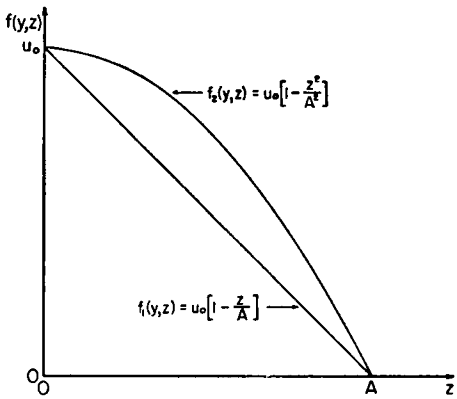

Figure 17. Initial pore pressure distribution, two dimensions.

The solution for the excess pore pressure in the clay soil for the case of the linear load function $f_1(y,z)$ is

$$u(z,y,T_v) = \frac{32u_0}{\pi^3} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n(2m+1)^2}$$

$$\sin \frac{n\pi}{2H} y \cos \frac{(2m+1)\pi}{2A} z\, e^{-\zeta T_v} \quad (77a)$$

$$\bar{u}(T_v) = \frac{256\bar{u}_0}{\pi^5} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{n^2(2m+1)^3} e^{-\zeta T_v} \quad (77b)$$

in which

$$\zeta = \frac{n^2\pi^2}{4} + \frac{(2m+1)^2\pi^2}{4}\phi \quad (77c)$$

When the load function is a simple quadratic form, such as $f_2(y,z)$, the solutions become:

$$u(z,y,T_v) = \frac{128u_0}{\pi^4} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{n(2m+1)^3}$$

$$\sin \frac{n\pi}{2H} y \cos \frac{(2m+1)\pi}{2A} z\, e^{-\zeta T_v} \quad (78a)$$

$$\bar{u}(T_v) = \frac{768\bar{u}_0}{\pi^6} \sum_{n=1,3,5}^{\infty} \sum_{q=1,3,5}^{\infty} \frac{1}{n^2q^4} e^{-\pi^2(n^2+q^2\phi)T_v/4} \quad (78b)$$

$$\zeta = \frac{n^2\pi^2}{4} + \frac{(2m+1)^2\pi^2}{4}\phi \quad (78c)$$

Linear Loading

The problems of linear loading were chosen as the exact analogues of the constant load problem. For a linear loading the rate, R, must be constant in time but is not necessarily constant in space. The conditions of constant load that were solved previously are the desired conditions at the end of the loading period. Thus, the loading conditions are space-dependent. Two rate functions were considered. The rate for a linear imposed excess pore pressure condition is:

$$R_1(z,t) = \frac{u_0}{t_0}[1-(z/A)] \quad (79a)$$

The second rate function selected was that for a quadratic loading function:

$$R_2(z,t) = \frac{u_0}{t_0}[1-(z/A)^2] \quad (79b)$$

The solution for the linear rate of loading, R_1, is:

$$u(z,y,T_v) = \frac{128u_0}{T_0\pi^5} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{1}{qn(2m+1)^2}$$

$$\sin \frac{n\pi}{2H} y \cos \frac{(2m+1)\pi}{2A} z$$

$$[1-e^{-(\pi^2q/4)T_v}] \quad (80a)$$

$$\bar{u}(T_v) = \frac{1028\bar{u}_0}{T_0\pi^7} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{qn^2(2m+1)^3}$$

$$[1-e^{-(\pi^2q/4)T_v}] \quad (80b)$$

in which

$$q = n^2 + (2m+1)^2\phi \quad (80c)$$

and

$$T_0 = \frac{C_y}{H^2} t_0 \quad (80d)$$

The solution for the quadratic loading function, R_2, is:

$$u(z,y,T_v) = \frac{512u_0}{T_0\pi^6} \sum_{n=1,3,5}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{qn(2m+1)^3}$$

$$\sin \frac{n\pi}{2H} y \cos \frac{(2m+1)\pi}{2A} z$$

$$[1-e^{-(\pi^2q/4)T_v}] \quad (81a)$$

$$\bar{u}(T_v) = \frac{3072\bar{u}_0}{T_0\pi^8} \sum_{n=1,3,5}^{\infty} \sum_{p=1,3,5}^{\infty} \frac{1}{wn^2p^4}$$

$$[1-e^{-(\pi^2w/4)T_v}] \quad (81b)$$

in which

$$q = n^2 + (2m+1)^2\phi \quad (81c)$$

$$w = n^2 + p^2\phi \quad (81d)$$

The solution to the construction loading problem can be ascertained by the proper combination of the constant load and the linear loading problems. The combination in the two-dimensional case has an exact analogue with the one-dimensional case.

Problems of variable permeability can also be solved by using the methods outlined for one-dimensional consolidation.

In order to keep the formulation within bounds of reasonable mathematical complexity, it becomes necessary to make some postulation on the horizontal and vertical components of the permeability. If the solution is constructed for the permeability transformation previously cited, the final solution is in terms of the vertical permeability. In order that the mathematical solution have a counterpart in reality, it is necessary to keep in mind that the variation in horizontal permeability obeys the same law as the change in vertical permeability. That is, the scale transformation is always constant.

THREE-DIMENSIONAL LOADING

The two-dimensional solution was developed specifically for a strip loading. Extension of the strip loading to a finite rectangular loading is a simple extension of the mathematics. The solutions for this type of loading are in terms of a triple series. It is questionable whether the computational effort involved is worthwhile, especially as other procedures may arrive at closely similar results with less effort.

By considering a circular imposed load, the consolidation problem maintains axial symmetry and, mathematically, reduces to two coordinate directions in a cylindrical coordinate system (ρ', z), as shown in Figure 18.

As in the two-dimensional case, the essential difference between this three-dimensional situation and the one-dimensional case is the factors involving radial pressure distribution. The problems of variable permeability follow as direct analogues of the one-dimensional consolidation problem considering time-dependent permeability. For this reason, only situations of constant permeability are considered here.

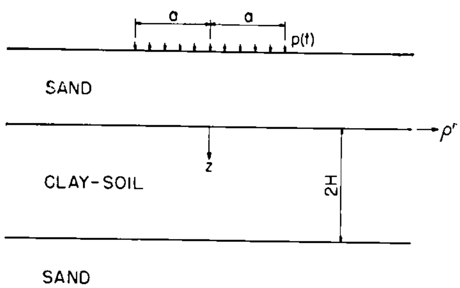

Figure 18. Double-drainage clay layer, circular loading.

The differential equation which governs the change of excess pore pressure is:

$$k_z \frac{\partial^2 u}{\partial z^2} + k_r \left[\frac{1}{\rho'} \frac{\partial u}{\partial \rho'} + \frac{\partial^2 u}{\partial \rho'^2} \right] + Q\gamma_w = m\gamma_w \frac{\partial u}{\partial t} \quad (82)$$

Eq. 82 is for a bi-dimensionally anisotropic material. To reduce it to an isotropic form, the usual scale transformation is made:

$$r = \rho' \sqrt{k_r/k_z} \quad (83a)$$

$$C_v \left[\frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right] + R = \frac{\partial u}{\partial t} \quad (83b)$$

$$C_v = k_z/m\gamma_w \quad (83c)$$

The boundary conditions that enable a solution to this differential equation are similar to the two-dimensional problem, as follows:

1. The top and bottom surfaces, being free draining, are always free of excess pore pressure, or

$$u(r,0,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq r \leqq A \quad (84a)$$

$$u(r,2H,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq r \leqq A \quad (84b)$$

2. For a surface loading of finite radius, the imposed pressure dissipates fairly rapidly in the radial direction. Thus, at some radius, A, this imposed excess pore pressure is zero, or

$$u(A,z,t) = 0 \qquad 0 \leqq t \leqq \infty \qquad 0 \leqq z \leqq 2H \quad (85)$$

This boundary condition is in terms of a perfectly general boundary radius, A. In consolidating a triaxial specimen, the radial boundary condition as indicated in Eq. 85 is applicable. Thus, the solution to this problem has quite broad applications.

3. The initial conditions imposed on

the problem are those of the initially imposed excess pore pressure:

$$u(r,z,0) = g(r,z) \qquad 0 \leqq z \leqq 2H \quad 0 \leqq r \leqq A \qquad (86)$$

Two types of initial conditions are considered. In all cases it is assumed that the vertical distribution of initial pore pressure is a constant. In the first case it also is assumed that the lateral distribution of initial excess pore pressure is constant; that is,

$$g_1(r,z) = u_0 \qquad (87a)$$

The second type of lateral initial pore pressure distribution is that of a dissipating pressure that follows a quadratic law:

$$g_2(r,z) = u_0[1-(r/A)^2] \qquad (87b)$$

4. The rate of loading, R, is a function of the time rate of imposing pore pressure at a point.

The general solution to this boundary value problem can be constructed for the general conditions previously outlined. This solution is:

$$u(r,z,t) = \frac{2}{A^2H}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \frac{H_0}{[J_1(m_i)]^2} J_0(m_i r/A) \sin\frac{n\pi}{2H}z\, e^{-C_v\zeta t} + \frac{2}{A^2H}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_0(m_i r/A)}{[J_1(m_i)]^2} \sin\frac{n\pi}{2H}z\int_0^t \Omega\, e^{-C_v\zeta(t-\tau)}\,d\tau \qquad (88a)$$

in which

$$H_0 = \int_0^A\int_0^{2H} r\, g(r,z) \sin\frac{n\pi}{2H}z\, J_0\left(m_i\frac{r}{A}\right)dz\,dr \qquad (88b)$$

$$\Omega = \int_0^A\int_0^{2H} r\, R(r,z,t) \sin\frac{n\pi}{2H}z\, J_0\left(m_i\frac{r}{A}\right)dz\,dr \qquad (88c)$$

and

$$J_0(m_i) = 0 \qquad (88d)$$

$$\zeta = (m_i/A)^2 + (n\pi/2H)^2 \qquad (88e)$$

Two classes of problems, both dealing with situations of constant permeability, are solved as follows:

1. Constant load.
2. Constant rate of loading.

Constant Load

The first constant load problem solved was for the case of a uniform initial excess pore pressure throughout the soil. This case has little validity in reality for a foundation. The constant load initial condition has use in that it establishes a basis for analyzing certain types of laboratory tests. There is at present some interest in the consolidation of soils under deviatoric stress conditions. In this test a triaxial specimen after pre-consolidation is loaded uniaxially or under a uniformly increasing uniaxial pressure. The former case is that of a constant initial excess pore pressure, u_0.

In cases of constant initial conditions, the solution is separable in terms of the radial and vertical components of consolidation (*16*). Thus, the solution can be written in the form:

$$u(r,z,t) = u_0 u_z(z,t)\, u_r(r,t) \qquad (89)$$

The solution for u_z is that of the one-dimensional consolidation for an initial excess pore pressure of unity.

The solution for the radial component is:

$$u_r(r,T_r) = 2\sum_{i=1}^{\infty}\frac{1}{m_i J_1(m_i)} J_0\left(m_i\frac{r}{A}\right)e^{-m_i^2 T_r} \qquad (90a)$$

in which

$$T_r = \frac{C_v}{A^2}t \qquad (90b)$$

and

$$J_0(m_i) = 0 \qquad (90c)$$

Measurements, in the laboratory and elsewhere, are most convenient in the vertical sense. Thus if

$$\psi = T_r/T_v = (H/A)^2 \qquad (91)$$

Then the complete solution can be determined:

$$u(r,z,T_v) = \frac{8u_0}{\pi}\left\{\sum_{i=1}^{\infty}\frac{J_0(m_i r/A)}{m_i J_1(m_i)} e^{-m_i^2\psi T_v}\right\}$$
$$\left\{\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2H}z\, e^{-(n^2\pi^2/4)T_v}\right\} \quad (92a)$$

$$\bar{u}(T_v) = \frac{32\bar{u}_0}{\pi^2}\left\{\sum_{i=1}^{\infty}\frac{1}{m_i^2} e^{-m_i^2\psi T_v}\right\}$$
$$\left\{\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} e^{-(n^2\pi^2/4)T_v}\right\} \quad (92b)$$

in which

$$J_0(m_i) = 0 \quad (92c)$$

The second constant load problem considered was for the initial condition expressed by Eq. 87*b*. This condition is an approximation of the stress conditions imposed on the layered soil mass. The solution to this problem is:

$$u(r,z,T_v) = \frac{32u_0}{\pi}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n\, m_i^3 J_1(m_i)}$$
$$J_0\left(m_i\frac{r}{A}\right)\sin\frac{n\pi}{2H}z\, e^{-\zeta T_v} \quad (93a)$$

$$\bar{u}(T_v) = \frac{256\bar{u}_0}{\pi^2}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2 m_i^4} e^{-\zeta T_v} \quad (93b)$$

$$J_0(m_i) = 0 \quad (93c)$$

$$\zeta = (n^2\pi^2/4) + m_i^2\psi \quad (93d)$$

Constant Rate of Loading

The first problem in constant rate of loading that was attacked was that of a constant rate throughout the consolidating soil mass, or

$$R = \frac{u_0}{t_0} \quad (94)$$

In terms of a triaxial test this condition is that of a uniform rate of applying the deviatoric stress.

The solution to this problem requires substitution of a constant, R, in the second sum of Eq. 88*a* and thus the evaluation of the integral in Eq. 88*c*. The final solution becomes:

$$u(r,z,T_v) = \frac{32u_0}{T_0\pi}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n m_i \zeta J_1(m_i)}$$
$$J_0\left(m_i\frac{r}{A}\right)\sin\frac{n\pi}{2H}z\left[1-e^{-(\zeta/4)T_v}\right] \quad (95a)$$

$$\bar{u}(T_v) = \frac{128\bar{u}_0}{T_0\pi^2}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2 m_i^2 \zeta}$$
$$\left[1-e^{-(\zeta/4)T_v}\right] \quad (95b)$$

$$\zeta = n^2\pi^2 + 4\psi m_i^2 \quad (95c)$$

$$J_0(m_i) = 0 \quad (95d)$$

The problem of a uniformly increasing stress is handled in a manner that is conventional in this paper. The loading rate is defined as

$$R = \frac{u_0}{t_0}[1-(r/A)^2] \quad (96)$$

The solution to this problem is:

$$u(r,z,T_v)$$
$$= \frac{128u_0}{T_0\pi}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n m_i^3 \zeta J_1(m_i)} J_0\left(m_i\frac{r}{A}\right)$$
$$\sin\frac{n\pi}{2H}z\left[1-e^{-(\zeta/4)T_v}\right] \quad (97a)$$

$$\bar{u}(T_v) = \frac{1824\bar{u}_0}{T_0\pi^2}\sum_{i=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2 m_i^4 \zeta}$$
$$\left[1-e^{-(\zeta/4)T_v}\right] \quad (97b)$$

$$\zeta = n^2\pi^2 + 4m_i^2\psi \quad (97c)$$

$$J_0(m_i) = 0 \quad (97d)$$

An extension of the foregoing solutions to problems of construction loading, and variable permeability can be treated in a manner analogous to the one- and two-dimensional cases. In performing these extensions the same factors that were inherent in the two-dimensional case must be considered here.

SAND DRAINS

The analysis of consolidation of soil by sand drains can be approached from two analytical points of view. The first approach, called the "free-strain" case (*17*), considers the flow of water to the drain, and thus the resulting surface displacement, to be non-uniform. Essentially the free-strain case corresponds to a situation of uniform surface loading. This condition forms one possible limit in terms of the behavior of the drained area. The other extreme is the equal strain case, wherein the settlement is uniform throughout. In actual practice the load redistribution of the sand blanket due to arching, with increasing consolidation and therefore settlement, will establish a strain pattern in the soil that is probably somewhere between equal- and free-strain. The work of Baron *(17)* and Richart *(5)* indicates that the differences between these two solutions is slight.

Mathematically, the equal-strain case leads to solutions in closed form which are much more readily computed than the free-strain solutions. In addition, the equal-strain case permits the development of non-linearities in the solution in a direct manner. For these reasons, the only case of sand drains developed specifically in this paper is that of equal strain.

The derivation of the differential equation of equal-strain sand drain theory follows the reasoning in the early portions of this paper. The equal-strain condition comes about by considering that the instanteous volume change of every element of the soil mass is equal in quantity to every other element. Thus, Eq. 18 becomes:

$$\nabla \cdot (k\nabla u) + Q\gamma_w = m\gamma_w \frac{\partial \bar{u}}{\partial t} \qquad (98)$$

Eq. 98 is amenable to exact solution because the right-hand side is a function of time only. The major difficulties that arise in exact solutions of complex problems are algebraic.

To illustrate the general methods of solution and to develop specific solutions of interest, certain specific problems are solved, as follows:

1. Constant load and variable permeability.

2. Variable load and constant permeability.

3. Variable load and variable permeability.

Two of these problems are solved for conditions in which no smear zone is present. The third is solved for the general case of a smear zone of finite thickness. In all cases the drain well is considered to be without well resistance, and only radial drainage is considered.

The formulation of the mathematical problem in Eq. 98 has been developed in dimensionless terms. In order to solve the exact problem the drain pattern must be considered. The original development of sand drain theory assumed that the drains were placed in an equilateral triangular pattern, as shown in Figure 19. This pattern, resulting in a circular zone of influence for each well, is the one used in this study.

The boundary details of a single well are shown in Figure 20 for cases of smear and no smear.

The theory of sand drains with smear considers the smear zone as one of com-

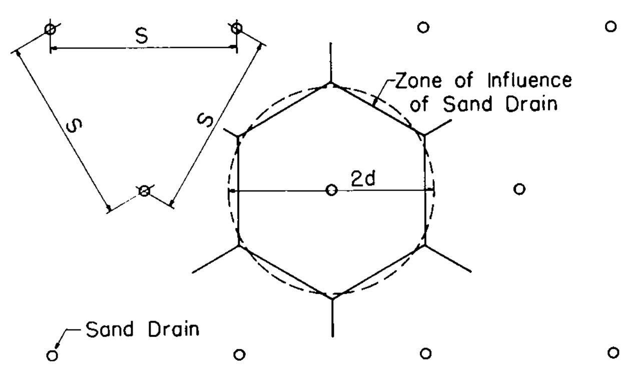

Figure 19. Sand drain pattern.

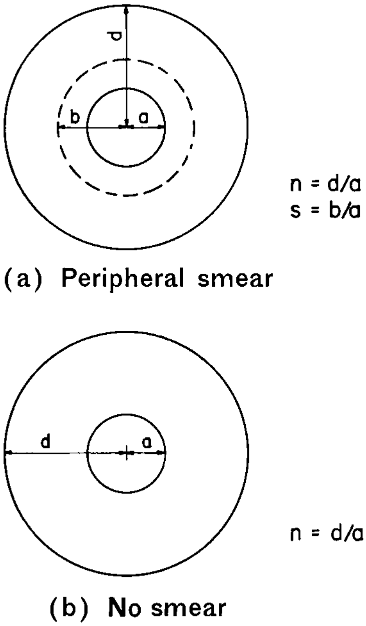

Figure 20. Sand drain details.

pletely remolded material that will be effectively incompressible. Thus, the pore pressure is determined by a set of simultaneous partial differential equations. In the smear zone:

$$\nabla \cdot (k''\nabla u'') + Q\gamma_w = 0 \quad a \leq r \leq b \quad (99)$$

Outside the smear zone the consolidation is governed by Eq. 98.

This is a problem of radial flow in which there is complete radial symmetry. Thus u'' and u are space functions of the radius only, and the problem is solved in polar coordinates.

The general boundary conditions which consider the existence of a smear zone are as follows:

1. The well has no resistance to drainage, or

$$u''(a) = 0 \qquad (100)$$

2. The well influence boundary is a surface through which no flow takes place, or

$$\frac{\partial u}{\partial r}(d,t) = 0 \qquad (101)$$

3. The interface between the smeared and unsmeared zones must have the same excess pore pressure and the same flow rates, or

$$u''(b) = u(b,t) \qquad (102a)$$

$$k''\frac{\partial u''}{\partial r}(b) = k\frac{\partial u}{\partial r}(b,t) \qquad (102b)$$

4. The initial condition depends on the type of rate function used. For constant load problems

$$\bar{u}(0) = \bar{u}_0 \qquad (103)$$

5. The rate of loading is always a function of time only. For the problems solved here this is merely a convenience, but for exact problems this condition becomes a necessity in order to develop a solution.

Constant Load, Variable Permeability

The problem of constant load and variable permeability solved in this paper is for a case of no smear.

The only additional working condition applied to the situation is that the coefficient of permeability is a function of time only. In other words, although the permeability may vary time-wise during the process of consolidation, it is constant space-wise. This condition is completely compatible with the basis of establishing equal-strain conditions. Because the volume changes of all elements are equal at a given instant in time, it can follow that the coefficient of permeability at a given instant of time is uniform throughout the mass.

This condition can be formulated as follows:

$$\bar{k}(t) = \alpha\bar{u}(t) + k_f \qquad (104)$$

The coefficient, α, will have the same numerical value as before, inasmuch as it depends only on the initial and final permeability states, which remain unchanged. Thus, the boundary value problem becomes:

$$D\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] = \frac{\partial \bar{u}}{\partial t} \qquad (105a)$$

$$D = \bar{k}/m\gamma_w \qquad (105b)$$

$$u(a,t) = 0 \qquad 0 \leq t \leq \infty \qquad (105c)$$

$$\frac{\partial u}{\partial r}(d,t) = 0 \qquad 0 \leq t \leq \infty \qquad (105d)$$

$$\bar{u}(0) = \bar{u}_0 \qquad a \leq r \leq d \tag{105e}$$

D is a function of time only and can be expressed in terms of two coefficients of consolidation, representing the final permeability and the gross difference in permeabilities. That is,

$$D = C_a(\bar{u}/\bar{u}_0) + C_f \tag{106a}$$

$$C_a = (k_0 - k_f)/m\gamma_w \tag{106b}$$

$$C_f = k_f/m\gamma_w \tag{106c}$$

The total solution to this problem takes a simple form:

$$\bar{u}(T_f) = \frac{\bar{u}_0 k_f}{k_f + \bar{k}_0(\bar{u}_0/\bar{u}_2 - 1)} \tag{107a}$$

in which

$$\bar{u}_2 = \bar{u}_0 e^{-[8/F(n)]T_f} \tag{107b}$$

$$T_f = (C_f/4d^2)t \tag{107c}$$

and

$$F(n) = \frac{n^2}{n^2-1}\ln(n) - \frac{3n^2-1}{4n^2} \tag{107d}$$

Thus, considering a linear variation in the average permeability, the change in excess pore pressure is modified by the extremities in the permeability. By knowing the permeabilities before and after consolidation, the theoretical excess pore pressure can be corrected to account for the variation in the permeability.

This problem can be extended in two areas. The first and logical extension is to include problems of smear. To satisfy the interface conditions of Eq. 102 the variations in the permeability along the interface must be considered. This presents no serious analytical problem but adds somewhat to the algebraic complexity of the solution; the procedure is not believed to be warranted at this time. However, when the proposed theory has been experimentally verified and additional information is forthcoming on the nature and extent of peripheral smear in sand drains, it would be a contribution to develop this more general theory.

The other extension is in terms of time-dependent loading. When a time-dependent loading term is included the resulting expression becomes needlessly complex, in light of the approximate methods which can be employed to equal advantage.

Variable Loading, Constant Permeability

The boundary value problem for a linear loading rate starting from zero load and constant permeability, can be formulated in terms of the general boundary value problem for a condition which includes peripheral smear:

$$C_h\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] + R = \frac{\partial \bar{u}}{\partial t} \qquad b \leq r \leq d \tag{108a}$$

$$C_h''\left[\frac{\partial^2 u''}{\partial r^2} + \frac{1}{r}\frac{\partial u''}{\partial r}\right] + R = 0 \qquad a \leq r \leq b \tag{108b}$$

$$u''(a) = 0 \qquad 0 \leq t \leq \infty \tag{108c}$$

$$u''(b) = u(b,t) \qquad 0 \leq t \leq \infty \tag{108d}$$

$$k_r''\frac{\partial u''}{\partial r}(b) = k_r\frac{\partial u}{\partial r}(b,t) \qquad 0 \leq t \leq \infty \tag{108e}$$

$$\frac{\partial u}{\partial r}(d,t) = 0 \qquad 0 \leq t \leq \infty \tag{108f}$$

$$\bar{u}(0) = 0 \qquad b \leq r \leq d \tag{108g}$$

$$R = \bar{u}_0/t_0 \qquad a \leq r \leq d \tag{108h}$$

The solution to this boundary value problem in terms of the average excess pore water pressure is:

$$\bar{u}(T_h) = \frac{\bar{u}_0}{8T_0}\left[F(n,s) - \frac{G(n,s)}{\chi}\right][1 - e^{-[8/F(n,s)]T_h}] \tag{109a}$$

in which

$$F(n,s) = \frac{n^2}{n^2-s^2}\ln\left[\frac{n}{s}\right] - \frac{3}{4} + \frac{s^2}{4n^2} + \theta\frac{n^2-s^2}{n^2}\ln(s) \tag{109b}$$

$$G(n,s) = \{s^2[1 - 2\ln(s)] - 1\}/2n^2 \tag{109c}$$

$$T_0 = (C_h/4d^2)t_0 \tag{109d}$$

$$T_h = (C_h/4d^2)t \tag{109e}$$

$$\chi = C_h''/C_h \tag{109f}$$

$$\theta = k_r/k_r'' \tag{109g}$$

If the peripheral smear zone does not exist, Eqs. 109 reduce to a simpler form:

$$\bar{u}(T_h) = \frac{\bar{u}_0 F(n)}{8T_0}\{1 - e^{-[8/F(n)]T_h}\} \tag{110a}$$

$$F(n) = \frac{n^2}{n^2-1}\ln(n) - \frac{3n^2-1}{4n^2} \qquad (110b)$$

Post-construction consolidation can be constructed from Eqs. 109 and 110 in the same manner as was done in the one-dimensional case. In the general case, where a peripheral smear zone exists, the post-construction consolidation is:

$$\bar{u}(T_h) = \frac{\bar{u}_0}{8T_0}\left\{F(n,s) - \frac{G(n,s)}{\chi}\right\} \left\{1 - e^{-[8/F(n,s)]T_0}\right\} e^{-[8/F(n,s)][T_h-T_0]} \qquad (111)$$

The terminology in Eq. 111 is the same as in Eqs. 109*b* through 109*g*.

The post-construction consolidation for no smear is:

$$\bar{u}(T_h) = \frac{F(n)\bar{u}_0}{8T_0}\{1 - e^{-[8/F(n)]T_0}\} e^{-[8/F(n)][T_h-T_0]} \qquad (112)$$

in which the terms were previously defined as in Eq. 110.

The basic component of both Eq. 110*a* and Eq. 112 is the exponent term. With a detailed knowledge of this term, specific solutions for either or both conditions can easily be constructed. To this end, the exponent is plotted *versus* the time factor, T_h in Figure 21. The time factor in this case is a dummy factor, in that it may take on one of three meanings, these being T_h, T_0, or $(T_h - T_0)$.

Variable Load, Variable Permeability

The problem of variable load and variable permeability was treated in combination by using the exponentional approximation presented previously in Eq. 32.

In adapting this basic relationship to equal-strain conditions that are favorable to solution, two additional working conditions are necessary, over and above the previously stated conditions.

The first condition is that the relationship between the average coefficient of permeability and the average excess pore pressure is the same as the relationship for point permeabilities as given in Eq. 104.

The constant k_f is identical in each case. This relationship is completely reasonable, because the basic condition governing the variation in permeability is formulated on porosity only.

Furthermore, k_f is considered to be independent of time and space. In this theory of equal strains the initial pore pressure is not uniform, but only the average initial pore pressure. However, since the final pore pressure is zero, there is no restriction on the final permeability. To make the theory and working conditions completely compatible, the a term must be redefined in terms of averages, as follows:

$$a = (\bar{k}_0 - k_f)/\bar{u}_0 \qquad (113)$$

The second additional working condition concerns the loading rate, R. To achieve the simplest form of solution, R is restricted to functions of time only. In reality this is a trivial restriction inasmuch as most, if not all, real problems concern an imposed loading due to the build-up of the fill on the surface. This type of loading can be considered as independent of space coordinates.

Using the working conditions specified, the boundary value problem for a linear build-up of fill, starting from a zero load, for a sand-drain system without smear is:

$$B_h\left[\frac{\partial^2 k}{\partial r^2} + \frac{1}{r}\frac{\partial k}{\partial r}\right] + a\,R = \frac{\partial \bar{k}}{\partial t} \qquad (114a)$$

$$k(a,t) = k_f \qquad 0 \leq t \leq \infty \qquad (114b)$$

$$\frac{\partial k}{\partial r}(d,t) = 0 \qquad 0 \leq t \leq \infty \qquad (114c)$$

$$\bar{k}(0) = k_f \qquad a \leq r \leq d \qquad (114d)$$

$$R = \bar{u}_0/t_0 \qquad a \leq r \leq d \qquad (114e)$$

The solution to this boundary value problem in terms of permeability is:

$$\bar{k}(t) = k_f + \frac{a\,d^2F(n)\bar{u}_0}{2B_ht_0}\{1 - e^{-[2B_h/d^2F(n)]t}\} \qquad (115)$$

in which $F(n)$ is defined as in Eq. 110*b*.

In terms of the average excess pore pressure the solution is:

$$\bar{u}(V_h) = \frac{\bar{u}_0F(n)}{8V_0}\{1 - e^{-[8/F(n)]V_h}\} \qquad (116a)$$

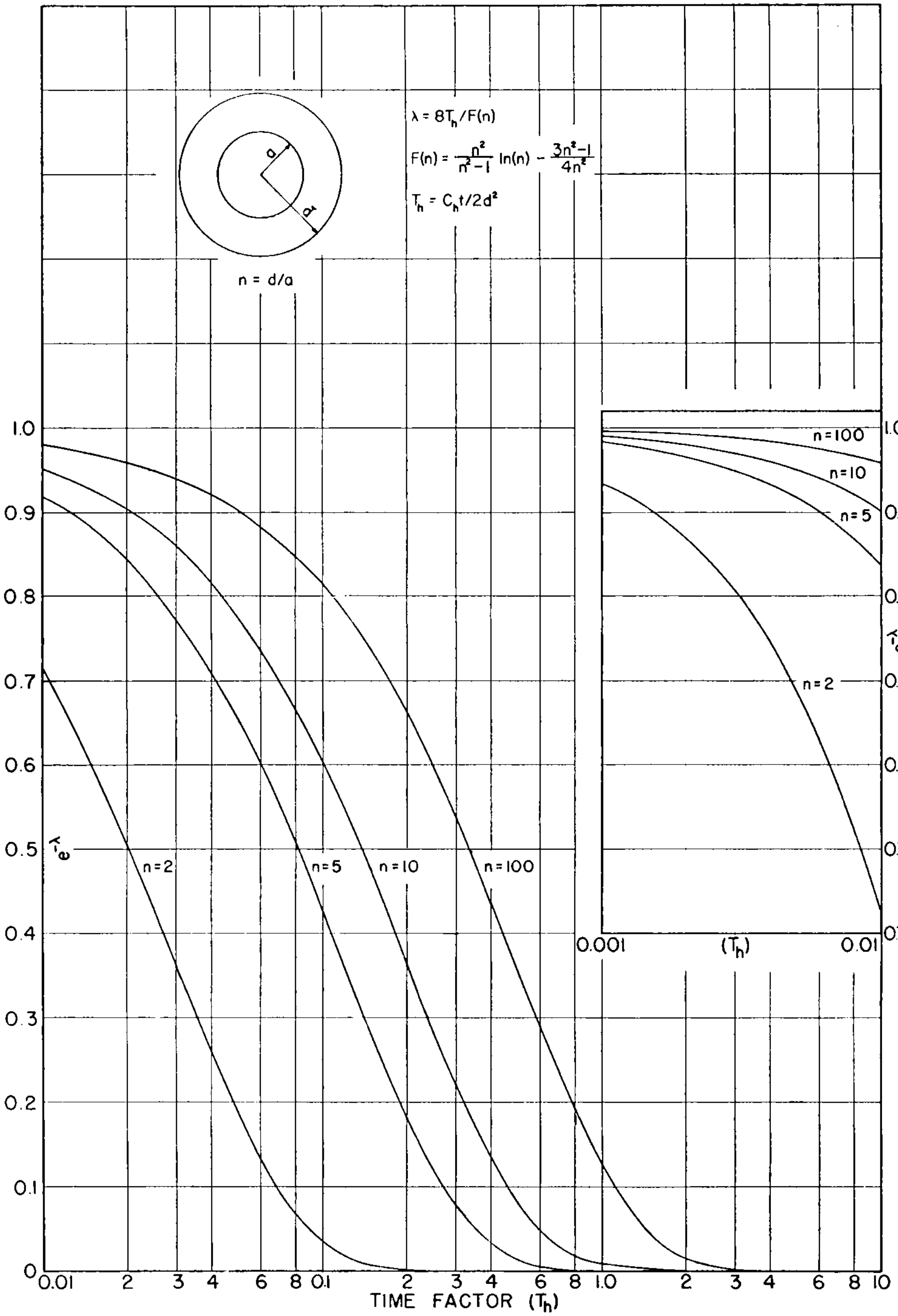

Figure 21. Exponents for equal-strain sand-drain problems, no smear, radial flow.

in which

$$V_0 = (B_h/4d^2)t_0 \qquad (116b)$$

$$V_h = (B_h/4d^2)t \qquad (116c)$$

$$B_h = (\bar{k}_0 - k_f)/m\gamma_w \ln(\bar{k}_0/k_f) \qquad (116d)$$

and $F(n)$ is defined as in Eq. 110b.

Specific solutions can be developed from the curves presented in Figure 21.

As in a previous case, the effects of peripheral smear were not considered. As before, the lack of consideration was not on the basis of lack of interest, but rather was due to the non-linear complexities inherent in the theory, which require solutions of such algebraic complexity as may preclude any practical usage.

The same arguments concerning the factor B_h apply for this case as for the one-dimensional case.

The situation of post-construction consolidation is formulated directly from Eqs. 116, as follows:

$$\bar{u}(V_h) = \frac{F(n)\bar{u}_0}{8T_0} \{1 - e^{-[8/F(n)]V_0}\} e^{-[8/F(n)][V_h - V_0]} \qquad (117)$$

The usage here is identical to the previously described usage.

CONCLUSIONS

This paper is a reasonably complete development of an extension to the theory of consolidation on two primary bases, as follows:

1. The permeability of the soil is considered to vary continuously with changes in the excess pore pressure.
2. The rate of loading is carried as an added factor in the theoretical problem.

It must be remembered that the conditions of permeability were all based on a postulated relationship between the coefficient of permeability and the excess pore pressure. Although there is every reason to believe that the proposed relationship is a correct one, the absolute validity of this relation will require experimental verification.

On the basis of the theory as herein presented, certain conclusions can be drawn, as follows:

1. The excess pore pressure dissipated during construction can be of invaluable aid in controlling post-construction settlements. Post-construction settlements can be materially reduced by artificially lengthening the construction period.
2. The coefficient of consolidation is, in actuality, dependent on the terminal permeabilities. This dependency is logarithmic; as a result, the use of mean permeabilities becomes seriously in error as the differences in terminal permeabilities increase.
3. The permeability is a function of the excess pore pressure. Thus, the coefficient of consolidation has an implied dependency on the initial excess pore pressure, and thus the magnitude of the load.
4. Consolidation of soils that are anisotropic with respect to permeability can be handled by a scale transformation.
5. Effects of variable permeability can be considered by simple curve fitting procedures where a laboratory curve is fitted segmentally to the theoretical curve.

It is hoped that future experimental work in the theory of consolidation will open the door to experimental proof of the foregoing theories. In particular, it is hoped that the approximations presented will be investigated in terms of their goodness of fit.

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APPENDIX — NOTATION

The following is a list of symbols used in this paper. Wherever possible the notation used is in conformance with the latest "Standard Definitions of Terms and Symbols Relating to Soil Mechanics," as proposed jointly by the American Society of Civil Engineers and the American Society for Testing Materials, Committee D-18.

In some instances it has been necessary to duplicate symbols in order to use accepted standards of notation. Where such duplication occurs the distinction is either obvious in terms of usage or widely separated by topic.

A = Effective half-width of consolidating mass in two-dimensional consolidation;

A = Effective radius of consolidating mass in axially symmetrical three-dimensional consolidation;

a = Half-width of loaded area in

two-dimensional consolidation;
a = Radius of loaded area in axially symmetric three-dimensional consolidation;
a = Radius of drain well;
B = Coefficient of consolidation permeability;
B_h = Coefficient of consolidation permeability for radial flow of sand drains in equal-strain case, using exponential approximation;
B_i = Coefficient of consolidation permeability during ith time increment of one-dimensional consolidation;
B_m = Coefficient of consolidation permeability during mth time increment of one-dimensional consolidation;
b = Outer radius of smear zone in sand drain theory;
C = Coefficient of consolidation for one-dimensional consolidation;
C_f = Coefficient of consolidation based on final permeability, sand drain theory;
C_h = Coefficient of consolidation in radial direction for sand drain theory, undisturbed zone;
C_h'' = Coefficient of consolidation in radial direction for sand drain theory, smear zone;
C_i = Coefficient of consolidation during ith time increment of one-dimensional consolidation;
C_m = Coefficient of consolidation during mth time increment of one-dimensional consolidation;
C_v = Coefficient of consolidation for vertical flow for axially symmetric three-dimensional flow and sand drains;
C_y = Coefficient of consolidation in vertical direction for two-dimensional consolidation;
C_a = Coefficient of consolidation for permeability difference, sand drain theory;
C_0 = Coefficient of consolidation based on initial permeability in one-dimensional consolidation;
D = Coefficient of consolidation based on linear variations of the volumetric average permeability, sand drain theory;
d = Radius of zone of influence of sand drain;
E = Error in the use of C as referred to B;
e = Base of natural logarithms;
F_0 = Fourier coefficient defined by Eq. 73b;
$F(n,s)$ = Sand drain dimension factor defined by Eq. 109b;
$F(n)$ = Sand drain dimension factor defined by Eq. 107d;
$f(z,y)$ = Initial excess pore pressure for two-dimensional consolidation;
$G(n,s)$ = Sand drain dimension factor defined by Eq. 109c;
$g(r,z)$ = Initial excess pore pressure for axially symmetric three-dimensional consolidation;
H = Half-thickness of one-dimensional consolidating layer;
H_0 = Fourier-Bessel coefficient defined by Eq. 88b;
h = Head of water at a given point in space of the consolidating mass and at a given instant of time;
i = Index;
$J_0(\ )$ = Bessel function of first kind and zero order;
$J_1(\ )$ = Bessel function of first kind and first order;
k = Coefficient of permeability as a point function of space and time;
$\bar{k}$ = Volumetric average coefficient of permeability as a function of time;
k_f = Coefficient of permeability at the end of the consolidation process;
k_i = Coefficient of permeability during the ith time increment;
k_r = Radial coefficient of perme-

ability in axially symmetric three-dimensional consolidation and in undisturbed portion of sand drains when permeability is a constant;

k_r'' = Radial permeability in smear zone for a sand drain;

k_x = Horizontal coefficient of permeability in two-dimensional consolidation when permeability is constant;

k_y = Vertical coefficient of permeability in two-dimensional consolidation when permeability is constant;

k_z = Vertical coefficient of permeability in axially symmetric three-dimensional consolidation when permeability is a constant;

k_0 = Coefficient of permeability as a space point function at zero time;

M = Substitution factor;

m = Modulus of volume change;

m = Summation index;

m_i = Root of Eq. 88d;

n = Summation index;

n = Ratio of radius of influence to well radius for sand drains;

n = Porosity;

n_0 = Reference porosity;

$\vec{n}$ = Unit normal vector;

p = Summation index;

$p(t)$ = Surface loading function;

p_0 = Reference surface load;

Q = Internal rate of head generation;

q = Substitution factor defined by Eq. 80c;

R = Rate of imposition of excess pore pressure;

r = Radial coordinate direction;

S = Surface element of arbitrary surface;

S = Spacing of drain wells;

s = Ratio of outer smear radius to well radius for sand drains;

T = Time factor for one-dimensional consolidation with constant permeability;

T' = Time factor for one-dimensional consolidation using constant permeability time increments;

T_f = Time factor for radial sand-drain flow considering final permeability;

T_r = Time factor for radial drainage in axially symmetric three-dimensional consolidation and constant permeability;

T_v = Time factor for vertical drainage in multi-dimensional consolidation and constant permeability;

T_x = Time factor for horizontal drainage in two-dimensional consolidation and constant permeability;

T_0 = End time factor; Time factor at end of loading period or at a particular reference time-constant permeability;

t = Time;

$\bar{t}$ = Time scaled to start at t_i;

t_i = Time at the ith time increment;

t_0 = Time at end of loading period, or a reference time;

u = Excess pore water pressure;

$\bar{u}$ = Volumetric mean excess pore water pressure;

u' = Imposed excess pore water pressure;

u'' = Excess pore water pressure in smear zone of a sand-drain system;

u_i = Excess pore water pressure during the ith time increment;

u_r = Excess pore water pressure in radial direction of axially symmetric three-dimensional consolidation;

u_z = Excess pore water pressure in vertical direction of axially symmetric three-dimensional consolidation;

u_0 = Initial excess pore water pressure for constant loading;

u_0 = Imposed excess pore water pressure at time t_0;

$\bar{u}_0$ = Volumetric average value of u_0;

u_1 = Reference excess pore pressure for harmonic loading;
$\bar{u}_2$ = Volumetric mean excess pore pressure for equal-strain, sand drains considering only the final permeability k_f;
V = Volume of unit element;
V = Time factor for one-dimensional consolidation and exponential approximation for varying permeability;
V_h = Time factor for radial flow to sand drains, using exponential approximation for varying permeability;
V_s = Volume of solids;
V_v = Volume of voids;
V_0 = End time factor for exponential approximation for varying permeability;
$\vec{v}$ = Velocity vector point function;
w = Substitution factor defined by Eq. 81d;
x = Horizontal rectangular coordinate direction;
y = Vertical rectangular coordinate direction;
z = Coordinate direction;
α = Modulus of permeability variation;
β = Proportionality between average permeability and porosity;
β_n = Substitution factor defined by Eq. 55;
γ_w = Unit weight of water;
δ = Total differential operator;
ϵ = Error in exponential approximation;
ϵ_{max} = Maximum ϵ;
ζ = Substitution factor;
η = Natural logarithm of μ;
Θ = Total mean stress;
$\bar{\Theta}$ = Effective mean stress;
θ = Ratio of k_r to k_r'';
Λ = Fourier coefficient defined by Eq. 73c;
μ = Ratio of k_0 to k_f;
ν = Proportionality between change of volume of voids and change in coefficient of permeability;
ρ = Exponential approximation constant;
ρ' = Radial coordinate direction for axially symmetric three-dimensional consolidation;
$\sigma(z)$ = Initial excess pore pressure for one-dimensional consolidation;
$\sigma_x, \sigma_y, \sigma_z$ = Normal stresses at a point in the soil mass, across surface whose orientation is defined by subscript coordinate;
τ = Variable of integration when limits are time;
ϕ = Ratio of vertical to horizontal time factors for two-dimensional consolidation;
χ = Ratio of the coefficients of consolidation, smear zone to unsmeared zone, for sand drain theory;
ψ = Ratio of radial time factor to vertical time factor for axially symmetric three-dimensional consolidation;
Ω = Substitution factor defined by Eq. 88c;
ω = Frequency of excess pore water pressure application for harmonic loading; and
∇ = Gradient operator.