# Warping Stresses and Deflections in Concrete Pavements: Part II

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A solution has been obtained for stresses and deflections of concrete slabs on ground subjected to temperatures and/or moisture contents which decrease linearly with the depth of the slab.

The derived equations were programmed with the aid of a Compiler and solutions were obtained from the Datatron 204 digital computer.

Nomographs are presented which give the ratios of the maximum stress to the modulus of rupture for a wide range of environmental gradients and for various combinations of slab sizes and thicknesses. The solutions are obtained for the critical condition which develops when the slab is only partially supported.

A 12- by 25-ft slab 8 in. thick, resting on a soil-cement subbase, was appropriately instrumented and subjected to temperature gradients under controlled conditions. Comparisons between computed and measured deflections confirm the validity of the new theory.

• TELLER and Sutherland (*5*) state in their discussion of pavement warping on the Arlington Test Road:

> The relation between the warping of the slab and the distribution of reactions is readily apparent. In the morning with the edges of the slab warped upward, the greatest (subgrade) pressure measured was in the interior, the pressure toward the edge being at its minimum at this time. During the day, as the upper surface of the slab expands, there is a complete relief of the high pressure in the interior region and a development of a maximum reaction near the edge, exceeding the maximum previously developed in the interior by 50 percent.
>
> Both the warping data and the subgrade pressure data suggest the possibility of an actual lifting of the central area from the subgrade as the edges warp downward.

Harr and Leonards (*1*) have extended Westergaard's analysis (*7*) for warped slabs to allow for the possibility that warping may result in only partial support of the slab by the foundation. A complete solution was obtained for the case where the temperatures and/or moisture contents increase linearly with the depth of the slab. More recently, Leonards and Harr (*2*) expanded their theory to include the effects of superimposed uniform loads, and axialsymmetric peripheral and point loads. In their earlier work, it was suggested that the basic theory be amplified to study the effects of warping and pavement support induced by temperatures and/or moisture contents which decrease with the depth of the slab.

In this paper, solutions have been obtained for the deflections, stresses and degree of support of concrete pavements subjected to temperatures and/or moisture contents which decrease linearly with the depth of the slab and are of sufficient magnitude to raise the central portion of the slab off its foundation.

Data from a controlled experiment on a 12- by 25-ft slab are reported and compared with predictions made from the new theory.

## THEORY

### *Assumptions*

1. Slab is of circular shape, homogeneous and isotropic with a free edge boundary.
2. Slab is supported on a homogeneous foundation.
3. Reaction between slab and foundation at any point of contact is proportional to the downward deflection of the foundation at that point.
4. Slab obeys Hooke's law.
5. Deflections of the slab are small in comparison with its thickness, which is constant.
6. External forces acting on the slab are those due to gravity only.
7. Slab is subject to ambient changes in temperature and/or moisture content which decrease linearly with the depth of the slab but are constant on all planes parallel to the upper and lower slab surfaces.

### *Formulation of Problem*

The specific solution developed herein pertains to the case where the temperature and/or moisture content decrease linearly with the depth of the slab; consequently, the edges of the pavement will tend to warp down. The slab is divided into two zones (Fig. 1) with the inner portion (zone 1) being a uniformly loaded slab unsupported by the foundation.

Conditions of continuity require that along the circle of radius $b$, the deflection, slope, bending moment, and shear of both the inner and outer portions be equal.

As the load and reactive forces are symmetrically disturbed about an axis perpendicular to the slab through its center, both the slope and the shear at the center of the slab must be zero.

The assumptions, boundary conditions, and symmetry may be summarized as follows (nomenclature is listed in Appendix A):

$$
\begin{aligned}
w_1'(0) &= 0\\
V_1(0) &= 0\\
w_1(\mathrm{b}) &= 0\\
w_2(\mathrm{b}) &= 0\\
V_2(\mathrm{a}) &= 0\\
M_2(\mathrm{a}) &= 0\\
w_1'(\mathrm{b}) &= w_2'(\mathrm{b})\\
M_1(\mathrm{b}) &= M_2(\mathrm{b})\\
V_1(\mathrm{b}) &= V_2(\mathrm{b})\\
w_1(0) &\leq 0
\end{aligned}
$$

Applying these conditions to the governing differential equations, equations for stresses and deflections in the slab are obtained (Appendix B).

## RESULTS

As was the case for upward warping, a prohibitive amount of work would be involved in eliminating the constants and obtaining an explicit expression for either the stresses or deflections in terms of the slab parameters. Hence, a high speed computer was used to obtain the desired information.

The derived equations were solved for deflections and maximum radial stresses

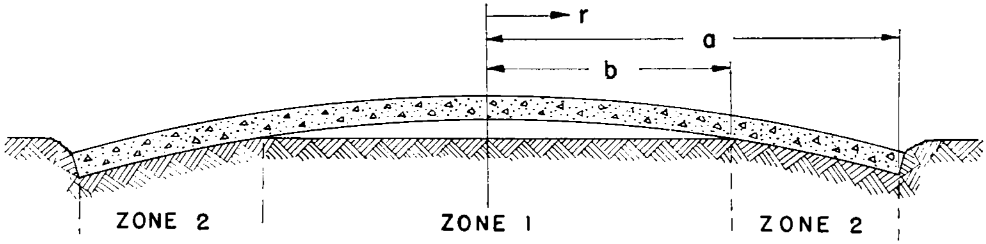



**Figure 1. Simplified diametral section of warped slab.**

for various combinations of the following:

$E = 3 \times 10^6, 4 \times 10^6, 5 \times 10^6$ psi
$a = 120, 150, 180, 210, 240$ in.
$h = 6, 8, 10, 12, 14, 18, 30, 48$ in.
$K = 100, 200, 400$ pci
$\alpha = 6 \times 10^{-6}$ in. per in. per °F
$\mu = 0.15$

Typical deflection curves for representative conditions are shown in Figure 2. Corresponding radial stresses as a function of radial distance from the center of the slab are shown in Figure 3.

Calculated stress relationships are shown in Figures 4a to 4i. These relationships are plots of the ratios of the maximum stress ($\sigma_0$) to the modulus of rupture ($M_r$) for effective temperature differences of 20, 30, and 40 F, and for various combinations of slab radius ($a$) and thickness ($h$). The use of these charts can best be illustrated by an example. For a stress ratio ($\sigma_0/M_r$) of 0.3, $E = 5 \times 10^6$ psi, $K = 100$ pci and, with an anticipated effective temperature difference of 30 F, (following the arrows in Fig. 4a) the curves show that among the various possible solutions are a 151-in. radius slab 8 in. thick, and 231-in. radius slab 18 in. thick.

As the curves for different values of $h$ (Fig. 4) followed the same trend, it was decided to calculate the curve for $h = 14$ in. only for a limited number of causes and estimate it for the remaining cases. For this reason the $h = 14$-in. curve is shown dotted in Figures 4.

The data for $a = 240$ in., $T = 30$ F are cross-plotted in Figure 5 to illustrate the effects of the quality of the concrete in the slab as characterized by the modulus of elasticity $E$.

*Examination of Assumptions*

The assumptions that the slab was circular, and had a free edge boundary, have already been examined (*1*) and found valid for unloaded slabs of a size of interest in the design of concrete pavements.

The no-support condition under the central portion of the slab (zone 1, Fig. 1) imposes a necessary but insufficient condition. In the prototype, the downward displacement of the center of the slab below the ground line would be resisted by the base. In the theoretical model it is possible for the central portion of the slab to be below the original ground line for two conditions: (a) when a large portion of the slab is unsupported and the slab is sufficiently flexible to permit its central portion to deflect beneath the original ground line, and (b) when the applied temperature gradient is insufficient to lift any part of the slab above the original ground line. Thus, it was necessary to specify that the deflection at the center of the slab be above the original reference line [$w_1$ (0) $\leq 0$]. This condition accounts for the limited range of some of the effective temperature curves in Figures 4a to 4i. For example: for a slab radius $a$ 240 in., $E = 3 \times 10^6$ psi, $K = 100$ pci (Fig. 4g), thicknesses of less than 14 in. and effective temperature differences between slab surfaces of 30°F or less, the deflection at the center of the slab is below the ground line and the analysis is invalid. For the same conditions with a large slab thickness ($h = 34$ in $\pm$), a temperature difference of 20°F is insufficient to cause any portion of the slab to warp up to the ground line.

*Comparison with Other Theoretical Data*

Comparing stresses in slabs subjected to upward warping with those that develop in downward warping, it was found that for equivalent slab parameters and effective temperature differences, the maximum stress in the former case is generally somewhat higher than in the latter. A typical comparison is shown in Table 1. It is apparent that the differences in maximum radial stresses are not great, and both downward and upward warping can be about equally destructive.

*Comparison with Observed Measurements*

A 12- by 25-ft by 8-in. portland cement concrete slab was constructed on a soil-cement subbase by the Portland Cement

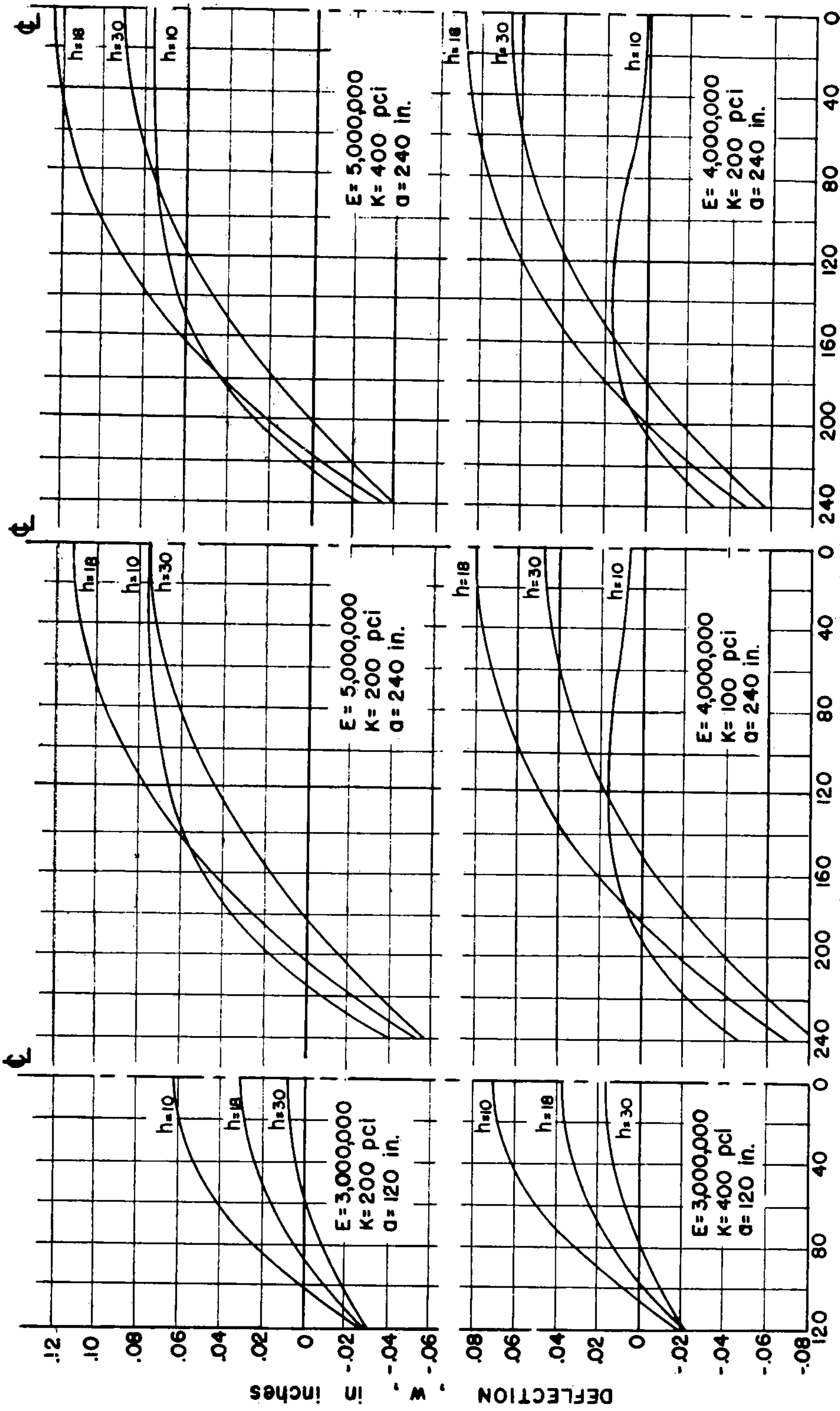



Figure 2. Representative deflection curves for an effective temperature difference of 30F between slab surfaces.

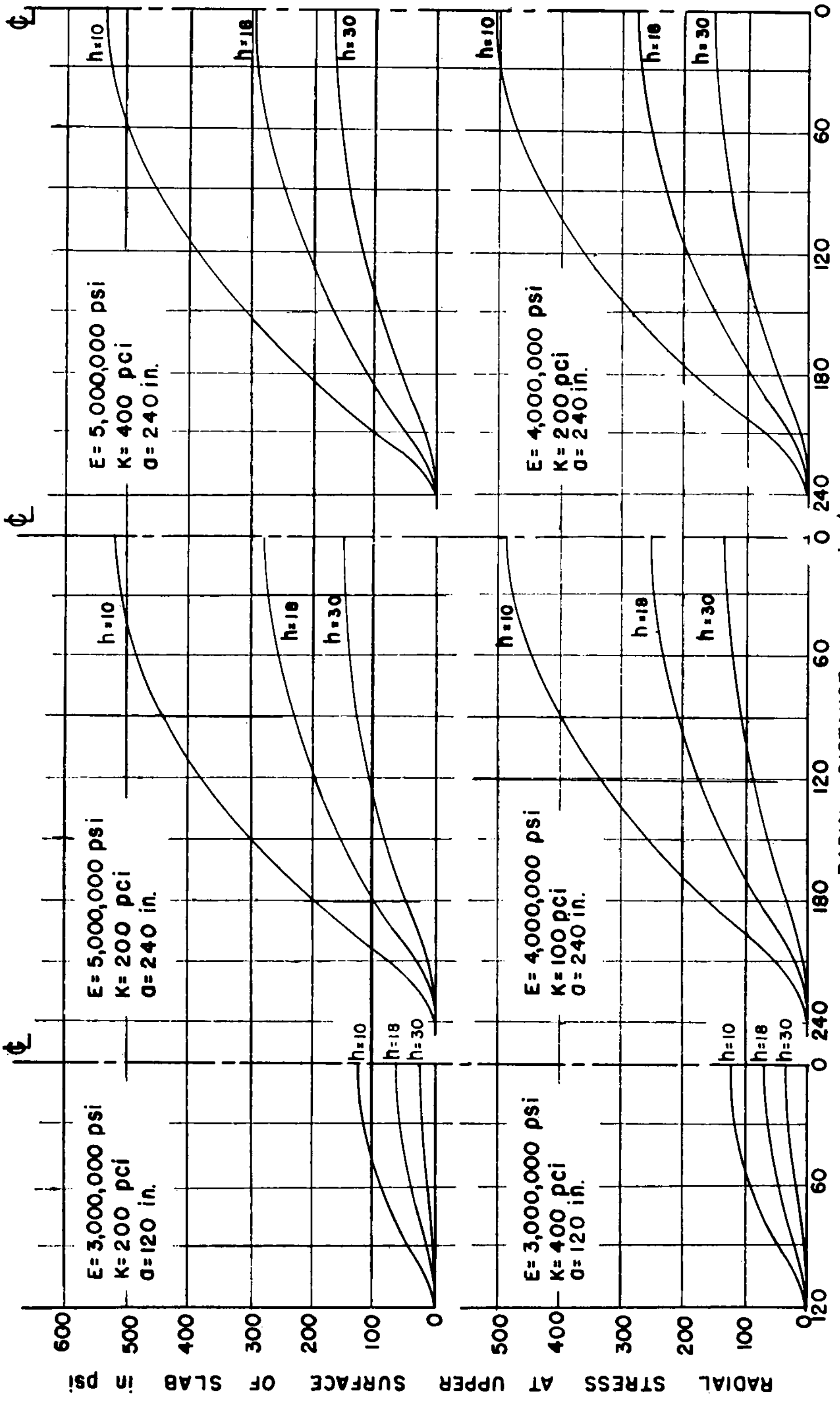
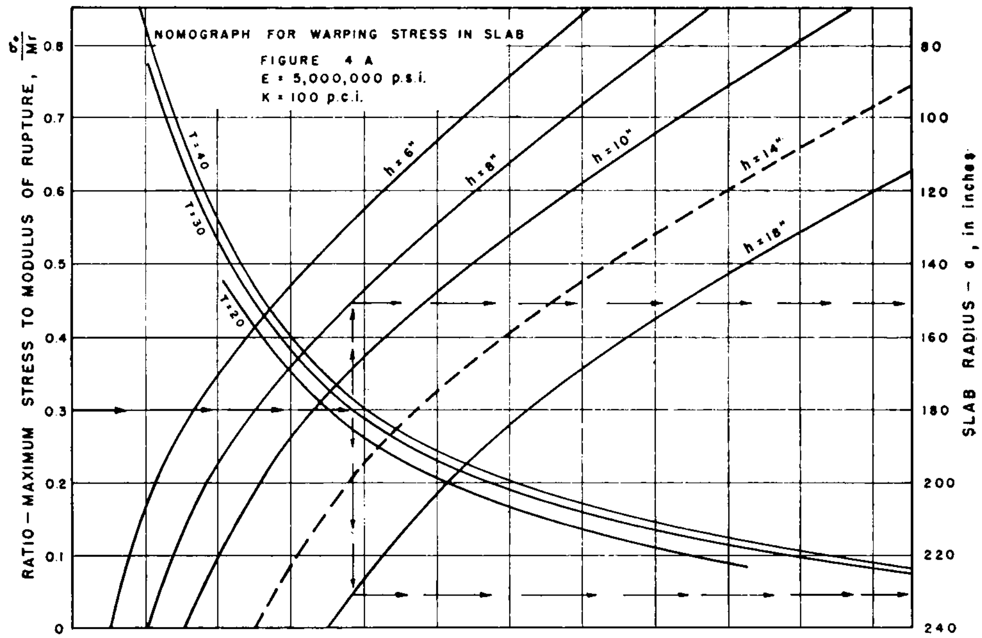



Figure 3. Representative radial stresses for an effective temperature difference of 30F between slab surfaces.



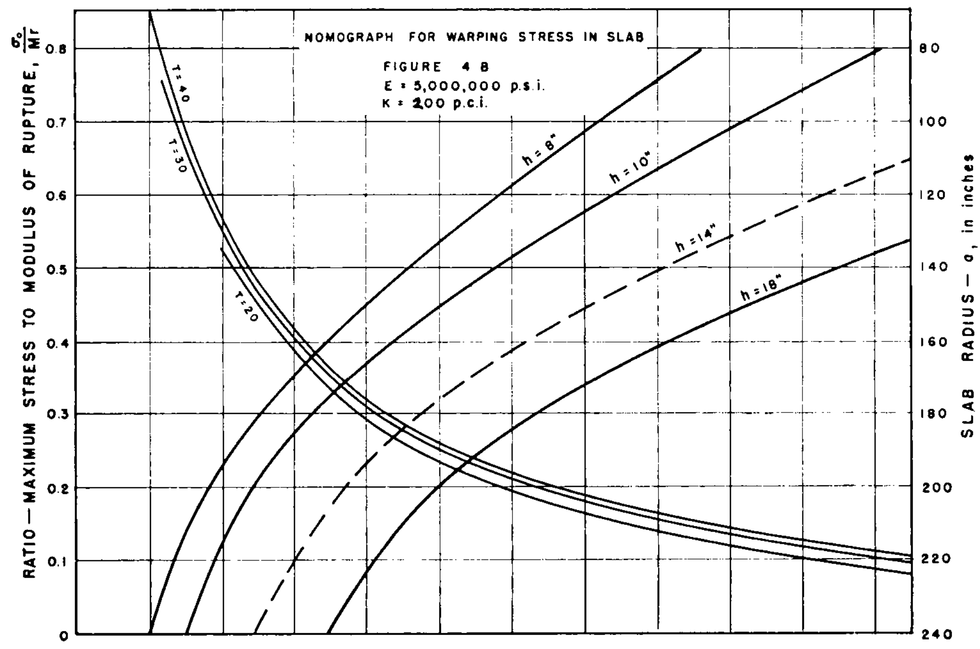
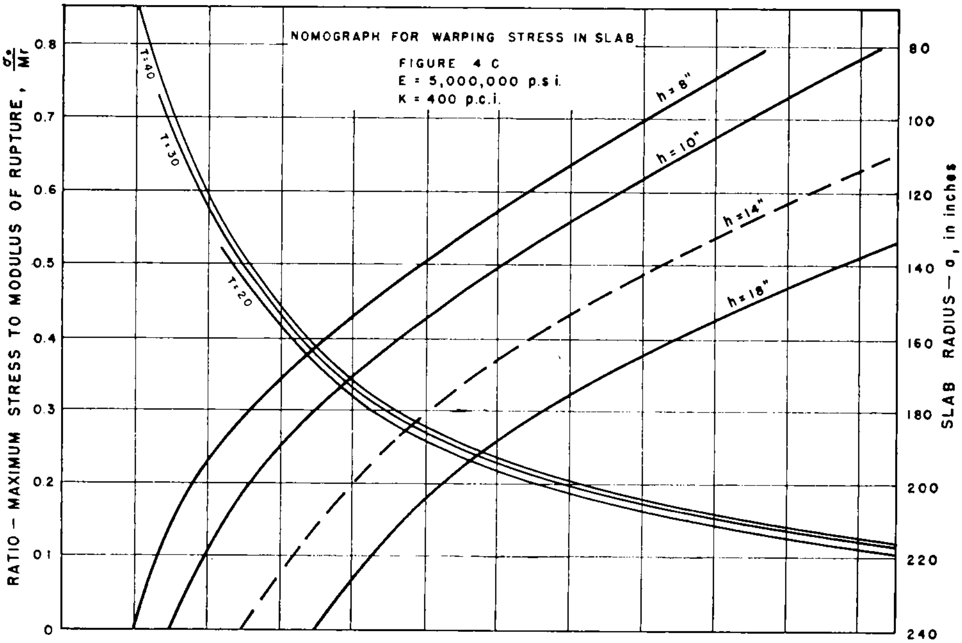




**Figure 4. Nomographs for warping stress in slab.**



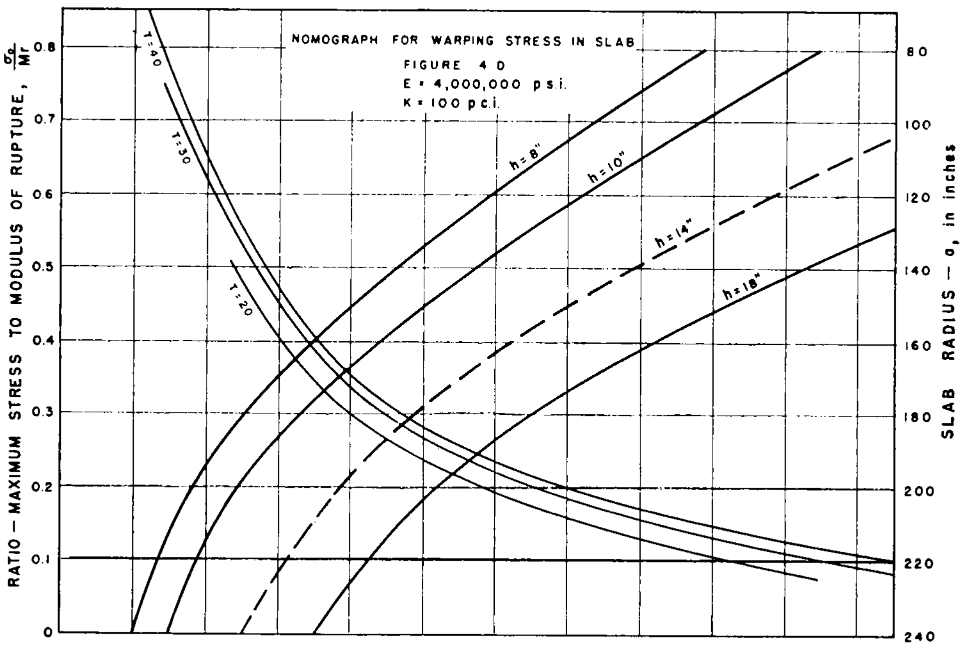
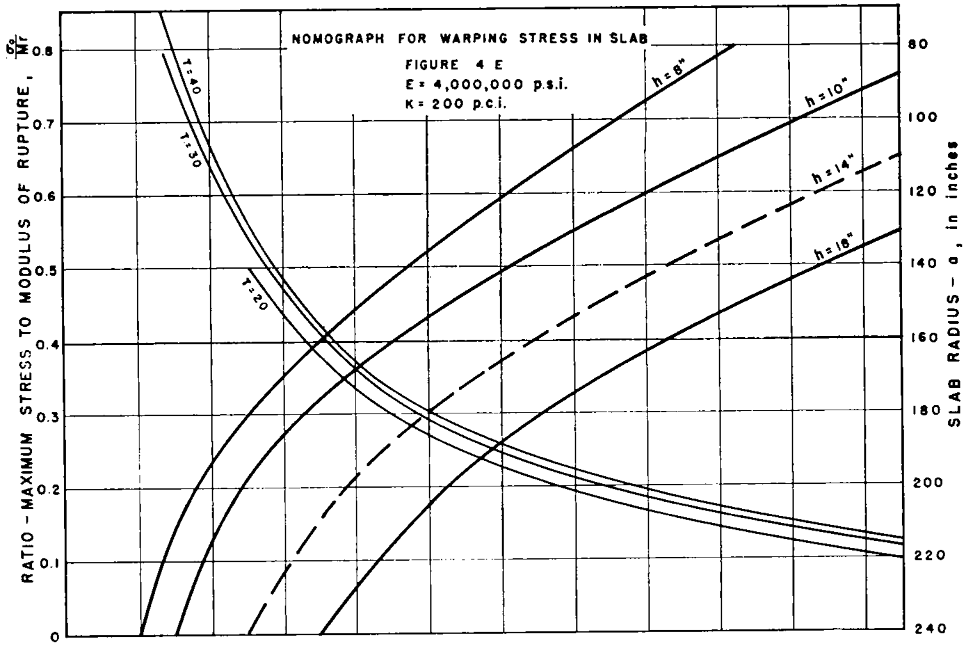




**Figure 4. (Continued).**



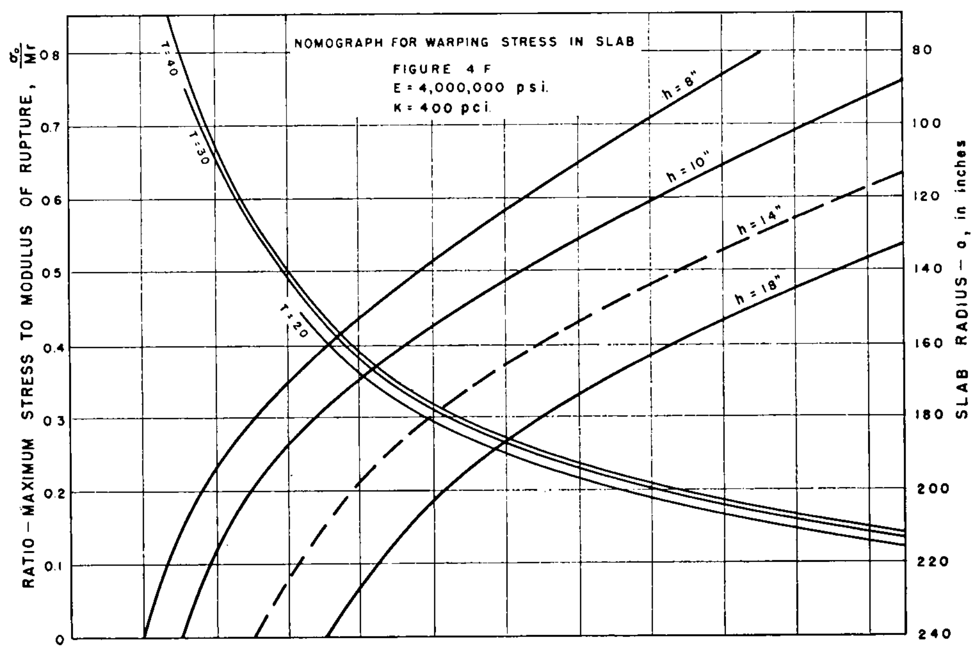
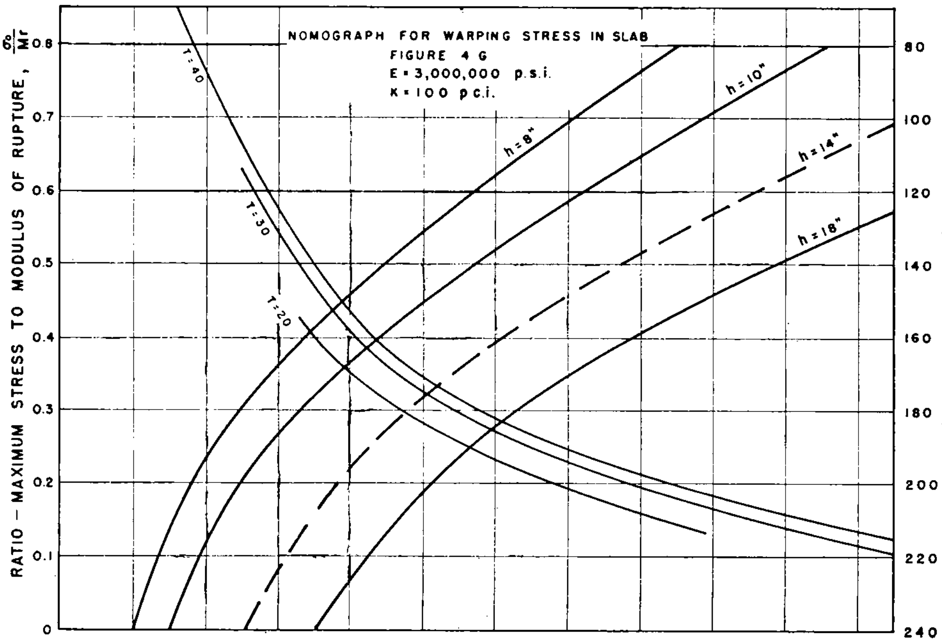




**Figure 4. (Continued).**



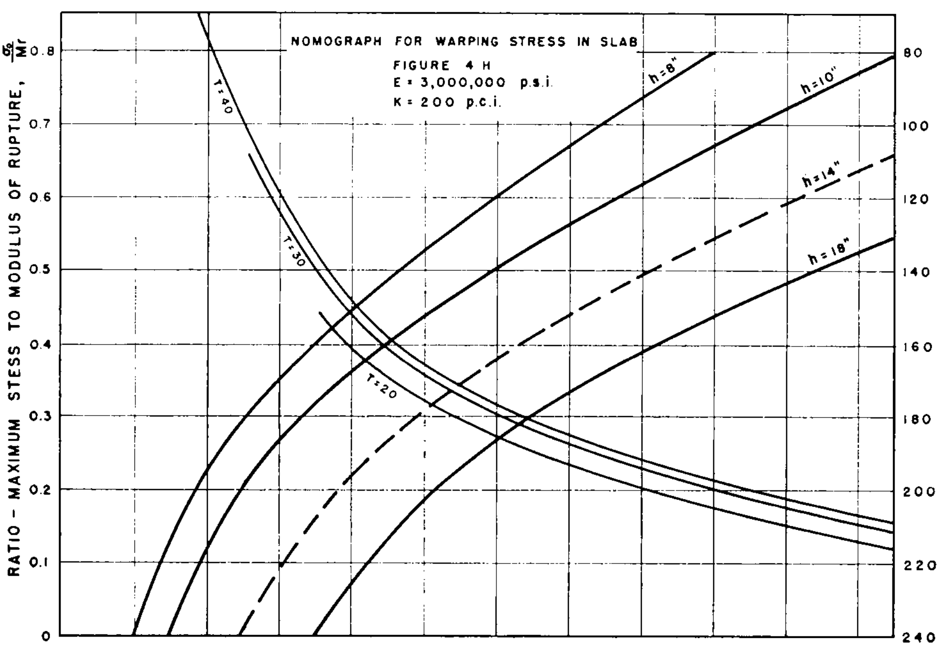




**Figure 4. (Continued).**

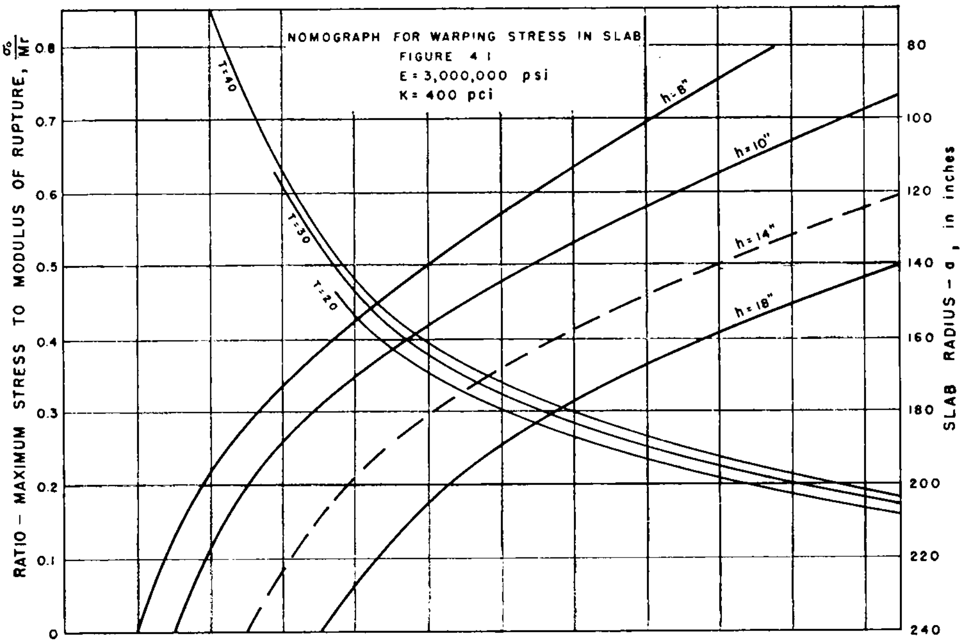



**Figure 4. (Continued).**

Association at their Skokie, Ill. laboratory. The slab was instrumented to measure temperatures, deflections, and the degree of unsupport, and was continuously inundated to prevent the development of moisture gradients. Temperature differences between slab surfaces were induced either with an ice slush or by circulating hot water over the surface of the slab. Temperatures and deflections were measured continuously until a linear temperature variation was obtained.

The sonic modulus of the concrete in the slab was evaluated to be 5,000,000 psi, and the initial modulus of subgrade reaction for the soil-cement subbase was measured to be between 360 and 400 pci.

Three types of deflection readings were taken: (a) movement of the slab relative to the subbase, (b) movement of the slab relative to a bridge above the slab surface supported on the floor of the laboratory, and (c) movement of the slab relative to deep rods anchored below the subbase.

TABLE 1

COMPARISON OF THEORETICAL DATA FOR VARIOUS SIZE SLABS: $E$ = 5,000,000 PSI, $K$ = 100 PCI, $T$ = 30 F

| $h$ | $a$ = 120 in. | | | $a$ = 180 in. | | | $a$ = 240 in. | | |
|---|---|---|---|---|---|---|---|---|---|
| (in.) | I[a] | II[b] | III[c] | I[a] | II[b] | III[c] | I[a] | II[b] | III[c] |
| 6 | 0.435 | 0.270 | 38.0 | 0.725 | 0.600 | 17.2 | 0.730 | — | — |
| 8 | 0.300 | 0.205 | 31.6 | 0.635 | 0.450 | 29.2 | 0.745 | 0.795 | −6.7 |
| 10 | 0.220 | 0.165 | 25.0 | 0.510 | 0.355 | 30.4 | 0.725 | 0.640 | 11.7 |
| 14 | 0.125 | 0.115 | 8.0 | 0.350 | 0.245 | 30.0 | 0.590 | 0.450 | 23.7 |
| 18 | 0.080 | 0.080 | 0.0 | 0.250 | 0.185 | 26.0 | 0.450 | 0.330 | 26.7 |

[a] $\sigma_0/M_r$ for upward warping.
[b] $\sigma_0/M_r$ for downward warping.
[c] Percent difference, $\left(\frac{I - II}{I}\right) \times 100$.

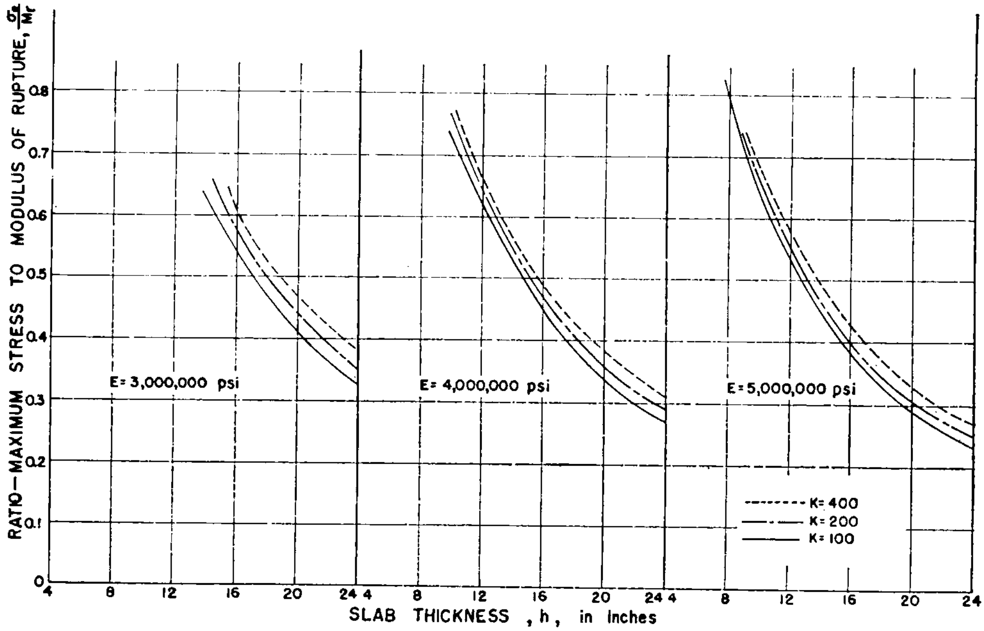



**Figure 5. Influence of modulus of elasticity; a = 240 in., T = 30F.**

Temperatures were measured at the top of the slab and at 1½-in. intervals across the depth of the slab. The location of the deflection measuring devices is shown in Figure 6. Figure 7 shows the instrumented slab during the test.

Figure 8 compares measured deflections (dotted curve) with the computed deflections (solid curve) for a circular slab with a radius equal to the length of the diagonal of the rectangular slab, a static modulus of elasticity of 4,000,000

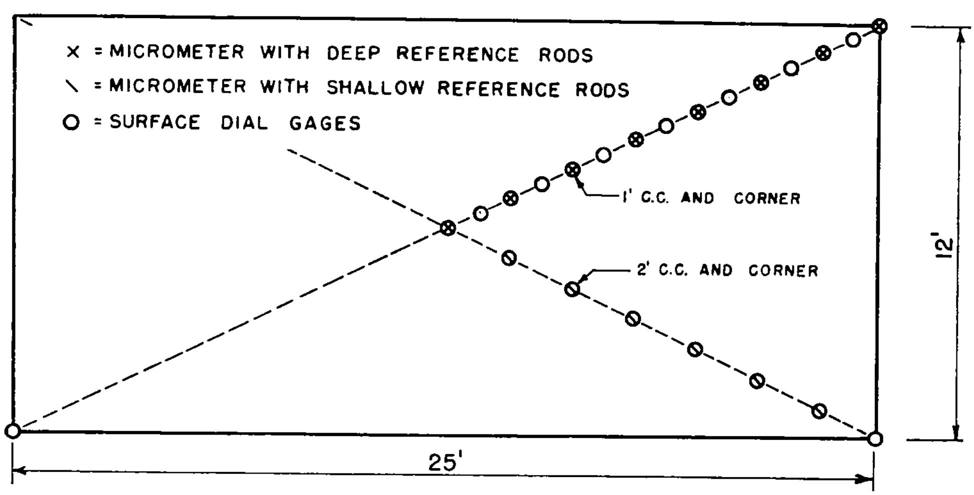



**Figure 6. Location of deflection measurements.**

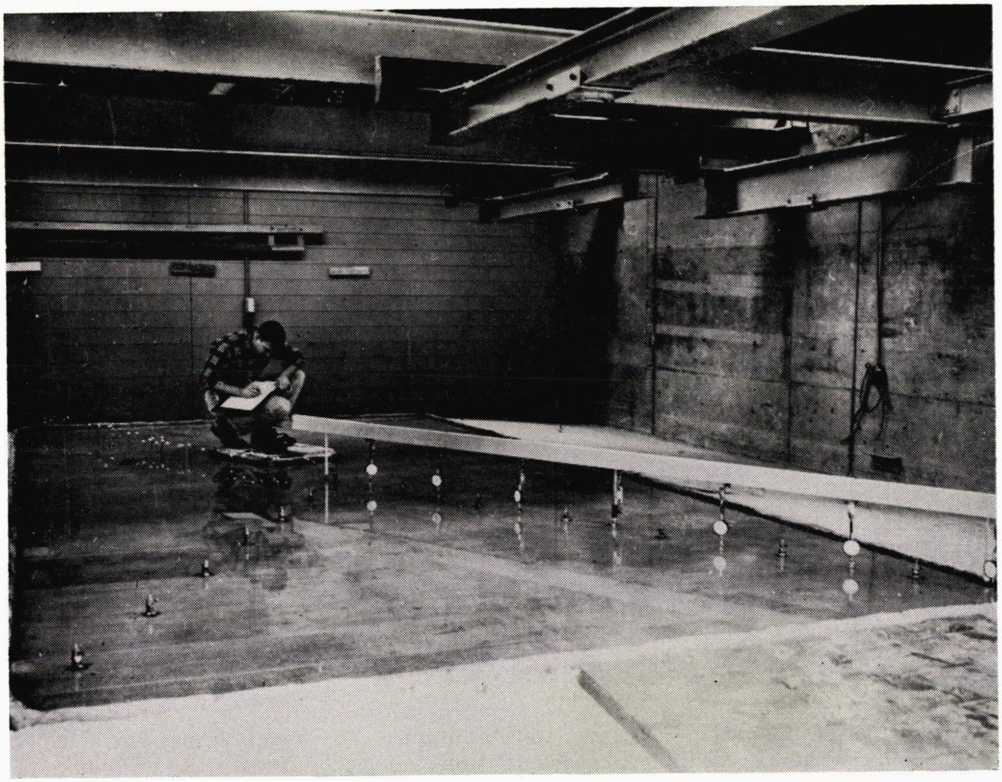


**Figure 7. Experimental slab, Portland Cement Association.**

psi, a modulus of subgrade reaction of 400 pci, and an effective linear temperature gradient, that decreases with the depth of the slab, of 20°F. Figure 9 compares measured deflections with the computed deflections of the same size circular slab with a modulus of elasticity of 4,000,000 psi, a modulus of subgrade reaction of 400 pci, and an effective linear temperature gradient, that increases with the depth of the slab, of 16°F.

The relation between the measured and theoretical deflection curves (Figs. 8 and 9) offers substantial evidence regarding the validity of the theory as applied to linear temperature gradients, in unloaded slabs with free edges.

*Effective Gradients*

It was shown by tests at Arlington, Va. (*5*) that temperature differences in the daytime (particularly in the summer after the sun had heated the upper surface of an 8-in. slab) reach a maximum of 4°F per inch of depth. Figures 4a to 4i show that effective temperature differences between slab surfaces greater than those generally encountered in practice are required to produce tensile stresses which approach the modulus of rupture of the slabs. Thus, damaging warping stresses are likely to develop only with a critical combination of temperature and moisture gradients.

## CONCLUSIONS

1. On the basis of the assumptions stated herein, the stresses, deflections and degree of support of concrete pavements subject to warping caused by linear ambient temperature and/or moisture variation can be calculated. Comparison with

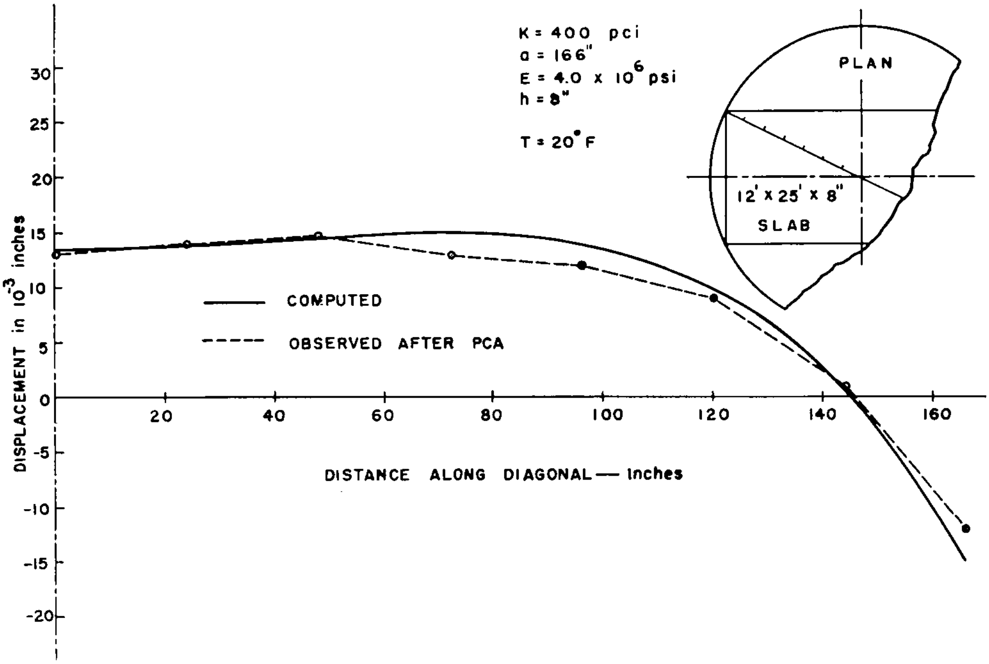



**Figure 8. Comparison between observed and computed deflections.**

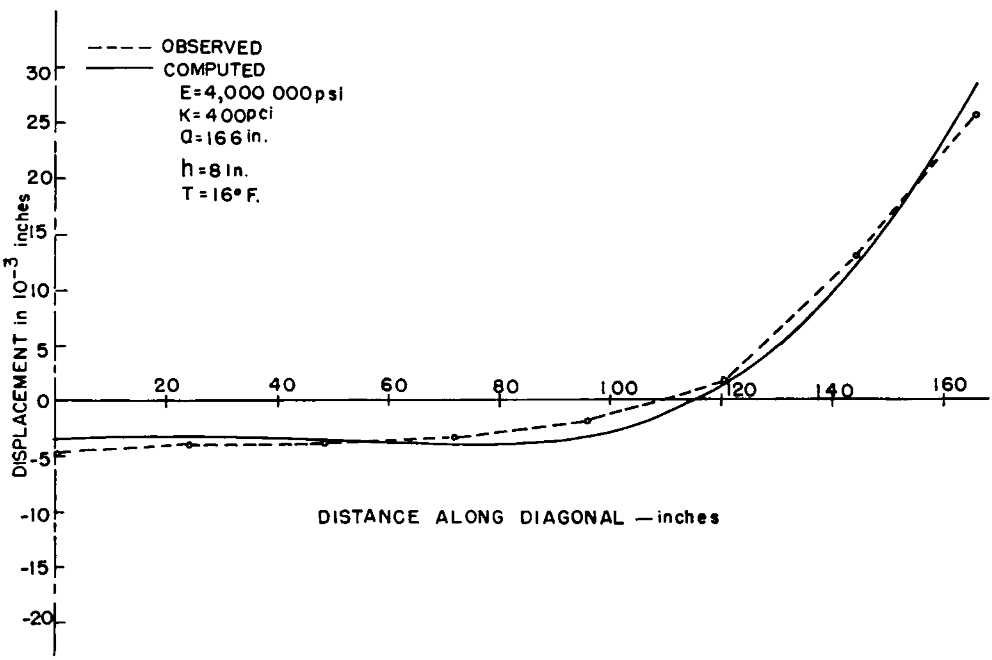



**Figure 9. Comparison between observed and computed deflections.**

measurements made under controlled conditions shows that the theory is valid under these circumstances.

2. Temperature and moisture gradients encountered in practice, when they so combine that their effects are additive, are capable of producing tensile stresses which may exceed the modulus of rupture of concrete highway slabs.

3. Rational design of concrete pavements requires that the effects of warping, including conditions of unsupport, be evaluated. This cannot be accomplished until data become available on the basis of which the critical moisture gradients that may develop in practice (and the volume changes that arise therefrom) can be evaluated.

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# APPENDIX A

NOMENCLATURE

Symbols in parentheses represent the dimensions of: $F$=force; $L$=length; and $t$=temperature.

$w$=deflection, positive in the downward direction $(L)$;
$q$=distributed load due to weight of slab $(F/L^2)$;
$K$=modulus of subgrade reaction $(F/L^3)$;
$p$=reaction of subgrade, $p=Kw(F/L^2)$;
$h$=slab thickness $(L)$;
$\mu$=Poisson's ratio;
$E$=Young's modulus;
$D=\frac{Eh^3}{12(1-\mu^2)}$=the flexural rigidity of the slab $(FL)$;
$l=\sqrt[4]{D/K}$=radius of relative stiffness $(L)$;
$\alpha$=linear coefficient of thermal expansion $(t^{-1})$;
$T$=temperature difference between upper and lower slab surfaces $(t)$;
$r$=radial distance $(L)$;
$a$=slab radius $(L)$;
$b$=radial distance to point of zero deflection $(L)$;
$\phi=\frac{a}{l}$;
$\beta=\frac{b}{l}$;
$\rho=\frac{r}{l}$;
$w'$=slope at point $r$;

$M(r) = -D\left[\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}+\alpha(1+\mu)\frac{T}{h}\right]$ = radial bending moment at point $r$ (*3*) $(FL/L)$;

$V(r) = -D\left[\frac{d}{dr}\nabla_r^2 w\right]$ = shear at point $r$ (*3*) $(F/L)$;

$\sigma(r) = -\frac{E_b}{2(1-\mu^2)}\left[\frac{M(r)}{D}\right]$ = normal radial stress at point $r$, positive denotes tension $(FL^{-2})$;

$\sigma_0$ = radial normal stress at center of slab $(FL^{-2})$;

$C_j$ = coefficient;

$Z_i(\rho)$ = Bessel functions;

$Z_i'(\rho)$ = first derivative of $Z_i(\rho)$;

$\nabla^2 = \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)$; and

ln = natural logarithm.

# APPENDIX B

The general differential equation for the deflection of a thin circular plate resting on an elastic foundation is given (*4*) as:

$$\nabla^4 w = \frac{q-p}{D} \qquad (1)$$

The solution of Eq. 1 was obtained by Schleicher (*4*) in the following form:

$$w = \frac{q}{K}(1+C_5Z_1(\rho)+C_6Z_2(\rho)+C_7Z_3(\rho)+C_8Z_4(\rho)) \qquad (2)$$

For the region where the plate is unsupported (Fig. 1) Eq. 1 reduces to:

$$\nabla^4 w = \frac{q}{D} \qquad (3)$$

The solution of Eq. 3 is given by Timoshenko (*6*) to be:

$$w = C_1 + C_2 \ln r + C_3 r^2 + C_4 r^2 \ln r + \frac{qr^4}{64D} \qquad (4)$$

## SOLUTION OF PROBLEM

Applying the conditions under the section on "Theory" to the governing equations for stresses and deflections in the slab

For $0 \leqq r \leqq b$

$$\sigma(r) = \frac{Eh}{2(1-\mu^2)}\left[2(\mu+1)C_3+\frac{qr^2}{16D}(3+\mu) - \alpha(1+\mu)\frac{T}{h}\right]$$

$$w_1(r) = (r^2-b^2)\left[C_3+\frac{q}{64D}(r^2+b^2)\right]$$

For $b \leqq r \leqq a$ or $\beta \leqq \rho \leqq \phi$

$$\sigma(\rho) = \frac{Eh}{2(1-\mu^2)}\left[\frac{q}{Kl^2}\left\{C_5\left(Z_2(\rho) - (1-\mu)\frac{Z_1'(\rho)}{\rho}\right) - C_6\left(Z_1(\rho)+(1-\mu)\frac{Z_2'(\rho)}{\rho}\right) + C_7\left(Z_4(\rho) - (1-\mu)\frac{Z_3'(\rho)}{\rho}\right) - C_8\left(Z_3(\rho)+(1-\mu)\frac{Z_4'(\rho)}{\rho}\right)\right\} - \alpha(1+\mu)\frac{T}{h}\right]$$

$$w_2(\rho) = \frac{q}{K}(1+C_5Z_1(\rho)+C_6Z_2(\rho)+C_7Z_3(\rho)+C_8Z_4(\rho))$$

where $C_3$, $C_5$, $C_6$, $C_7$, $C_8$, and $T$ may be obtained from the following equations.

$$C_5Z_1'(\beta)+C_6Z_2'(\beta)+C_7Z_3'(\beta)+C_8Z_4'(\beta) = \frac{Kl}{q}\left(2bC_3+\frac{qb^3}{16D}\right)$$

$$C_5Z_1(\beta)+C_6Z_2(\beta)+C_7Z_3(\beta)+C_8Z_4(\beta) = -1$$

$$\left\{Z_2(\beta)-(1-\mu)\frac{Z_1'(\beta)}{\beta}\right\}C_5-\left\{Z_1(\beta)+(1-\mu)\frac{Z_2'(\beta)}{\beta}\right\}C_6+\left\{Z_4(\beta)-(1-\mu)\frac{Z_3'(\beta)}{\beta}\right\}C_7$$
$$-\left\{Z_3(\beta)+(1-\mu)\frac{Z_4'(\beta)}{\beta}\right\}C_8=\frac{Kl^2}{q}\left[2(\mu+1)C_3+\frac{qb^2}{16D}(3+\mu)\right]$$

$$C_5Z_2'(\beta)-C_6Z_1'(\beta)+C_7Z_4'(\beta)-C_8Z_3'(\beta) = \frac{b}{2l}$$

$$C_5Z_2'(\phi)-C_6Z_1'(\phi)+C_7Z_4'(\phi)-C_8Z_3'(\phi) = 0$$

$$\left\{Z_2(\phi)-(1-\mu)\frac{Z_1'(\phi)}{\phi}\right\}C_5-\left\{Z_1(\phi)+(1-\mu)\frac{Z_2'(\phi)}{\phi}\right\}C_6+\left\{Z_4(\phi)-(1-\mu)\frac{Z_3'(\phi)}{\phi}\right\}C_7$$
$$-\left\{Z_3(\phi)+(1-\mu)\frac{Z_4'(\phi)}{\phi}\right\}C_8-\frac{Kl^2}{q}\left\{\frac{\alpha(1+\mu)}{h}\right\}T=0$$

The problem was programmed using the Purdue Compiler for the Datatron 204 digital computer.

The Bessel functions $Z_1(\rho)$ and $Z_2(\rho)$ and their first derivatives were computed from standard library subroutines. The Hankel functions $Z_3(\rho)$ and $Z_4(\rho)$ and their first derivatives were interpolated from tabular values stored in the main memory of the computer.

The main program was written with $b$, the radial distance to the point of zero deflection, as the independent variable; and $T$, the effective temperature difference between the slab surfaces, as the dependent variable. (The program is on file at the Statistical Laboratory, Purdue University, Lafayette, Ind.) Thus, by introducing a set of parameters ($K$, $a$, $E$, and $h$), the computer would solve for $T$ corresponding to a given value of $b$, and then print out the stress and deflection at equal radial increments across the slab. Repeating the cycle for increasing (or decreasing) values of $b$, the stresses and deflections were obtained for a wide range of temperatures.