THEORY OF FLOW THROUGH SHORT TUBES WITH SMOOTH AND CORRUGATED SURFACES AND WITH SQUARE EDGED ENTRANCES

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The University of Iowa tests on the flow of water through culverts, made some twenty years ago, are well known for the extensiveness of the measurements undertaken. The main purpose of the investigation was the establishment of the formulas of discharge for culverts. The investigation proposed the relations

$$Q = \frac{A\sqrt{2gH}}{\sqrt{1 + 0.31D^{0.5} + \frac{0.026L}{D^{1.2}}}} \qquad (1)$$

and

$$Q = \frac{A\sqrt{2gH}}{\sqrt{1 + 0.16D^{0.6} + \frac{0.106L}{D^{1.2}}}} \qquad (2)$$

as the formulas of discharge for concrete and corrugated metal pipes, respectively, the pipes having square-edged entrances and flowing full. In these expressions Q is the discharge, A the cross-sectional area of the pipe, H the difference in the elevations of the water surfaces at the two ends of the pipe, L the length, and D the diameter. Quantities are measured in feet and second units. Although the experiments were made with culverts of small length and size, the investigations suppose that the formula are equally valid for culverte of any length and size.

That these formulas faithfully represent the discharges actually met with in the tests of the particular culverts employed there can be no doubt. On the other hand, the validity of the formulas can certainly be questioned for culverts of greater length and size. It is the particular structure of the formulas proposed which invites criticism. Under the radical sign of the denominators of these formulas are found three terms which in their order are associated with the loss of kinetic energy at the exit end, the loss at entrance, and the loss throughout the entire length of the tube. The criticisms to be made are in regard to the forms of the last two terms. First, the dimensions of these terms are not correct. Second, the particular exponents of the diameters in these terms imply that both the entrance loss and the so-called Manning's n for the entire culvert length vary with the culvert size. With the increase of diameter, these two quantities increase. There is, now, no hydrodynamical basis for this implication to be true and, therefore, the apparent contradictions involving the formula proposed require an examination.

In dealing with a short tube, it is better to divide it into three segments. See Figure 1. There is first the entrance segment where the losses occur in the surface of discontinuity of the gradually vanishing vena contracta. Here the losses occur in the body of water and the frictional shear at the wall is negligible. According to this view the loss is independent of the size of tube or the character of the surface. We shall denote the length of the segment by L_o. Secondly, there is the boundary layer segment, i.e. the segment in

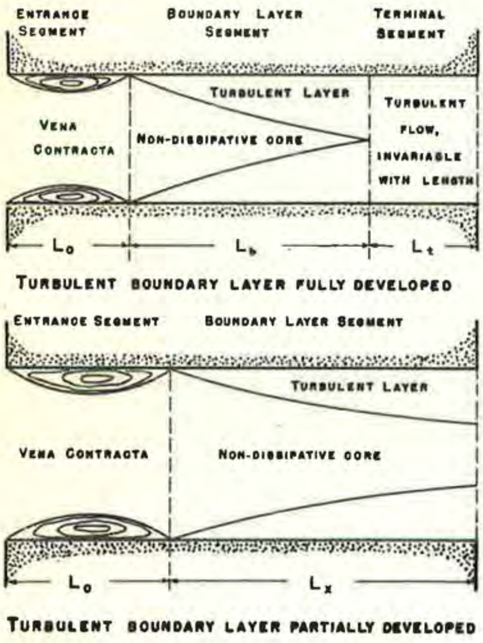

Figure 1. The Characteristic Flow Segments in a Short Pipe

which the turbulent boundary layer develops. At the upstream end of the segment, the thickness of the boundary layer is negligible and at the downstream end, when the tube is of sufficient length to assure the full development of the layer, the thickness equals the radius of the pipe. The local friction at the wall is large where the thickness of the layer is small. In the boundary layer segment, wall friction commences with a large value and is gradually reduced to a limiting value with distance. The central core outside of the boundary layer is non-dissipative and there the velocities and the pressures conform to the law of Bernoulli. For the full development of the layer a distance L_b will be required. Finally, if the length of the pipe exceeds $L_o + L_b$, there is the third segment, the terminal segment, where the velocity pattern does not change with distance along the pipe axis. We shall denote the length of the terminal segment by L_t. Now if the pipe is not sufficiently long, the length of the segment in which the turbulent layer develops will be less than L_b, and the corresponding length in that case may be denoted by L_x, where $L_x < L_b$.

THE THEORETICAL FORM OF DISCHARGE FORMULAS

Suppose that a tube of diameter D and length L connects two reservoirs. Let H be the difference in the elevations of the water surface in the two reservoirs. The Weisbach formula corresponding to the case is

$$H = \left(1 + f_e \frac{L}{D}\right) \frac{V^2}{2g} \qquad (3)$$

and the discharge will be given by

$$Q = \frac{A\sqrt{2gH}}{\sqrt{1 + f_e \frac{L}{D}}} \qquad (4)$$

Here, in these two expressions, f_e is the effective resistance coefficient. The proper evaluation of f_e requires that the losses are first individually determined for the three segments above mentioned.

Consider the case for which $L < L_o + L_b$. Denoting the loss in the entrance segment by H_o and in the boundary layer by H_b,

$$H = H_o + H_b + \frac{V^2}{2g}$$

In terms of the velocity head

$$H_o = f_o \frac{L_o}{D} \cdot \frac{V^2}{2g}$$

and

$$H_b = f_b \frac{L_x}{D} \cdot \frac{V^2}{2g}$$

where f_o is the coefficient of friction in the entrance segment and f_b is the coefficient of friction in the boundary layer segment. Thus

$$H = \left(1 + f_o \frac{L_o}{D} + f_b \frac{L_x}{D}\right) \frac{V^2}{2g} \qquad (5)$$

or comparing with equation 3,

$$f_e = f_o \frac{L_o}{L} + f_b \frac{L_x}{L} \qquad (6)$$

Consider next the case for which $L > L_o + L_b$, that is when

$$L = L_o + L_b + L_t$$

We now add to the right hand member of equation (5) the term H_t,

$$H_t = f \frac{L_t}{D} \frac{V^2}{2g}$$

where f is the coefficient of resistance in the terminal segment. Comparing the resultant expression of H with equation (3), it is seen that

$$f_e = f_o \frac{L_o}{L} + f_b \frac{L_b}{L} + f \frac{L_t}{L} \qquad (7)$$

Entering the expressions for the effective coefficient of friction from equation (6) or equation (7) into equation (4), we have for the discharges,

$$Q = \frac{A\sqrt{2gH}}{\sqrt{1 + f_o \frac{L_o}{D} + f_b \frac{L_x}{D}}} \qquad (8)$$

when $L = L_o + L_x$, and

$$Q = \frac{A\sqrt{2gH}}{\sqrt{1 + f_o \frac{L_o}{D} + f_b \frac{L_b}{D} + f \frac{L_t}{D}}} \qquad (9)$$

when $L = L_o + L_b + L_t$.

These are the theoretical discharge formulas. Comparing the general form of these expressions with the forms of the Iowa formulas, equations (1) and (2), the following facts are revealed. First, there is no unique formula which will express the discharge through short and long tubes with equal correctness. Secondly, by introducing the concept of the varying resistance in the laminar boundary layer, one dispenses with the idea of variable Manning's n. Actually, it is supposed that the variation of resistance in the critical initial part of a short tube is brought about not by the variation of the surface characteristics, since these are invariable, but by the gradual change in the thickness of the developing turbulent boundary layer.

THE EVALUATION OF THE EFFECTIVE COEFFICIENT OF RESISTANCE

To evaluate the effective coefficient of resistance f_e, the quantities f_o, L_o, L_b or L_x, f_b and f must be first known.

The examination of entrance losses in the Iowa experiments shows that for concrete and corrugated metal pipes with square-edged entrances, the entrance length L_o equals 3.5D and the coefficient of friction is

$$f_o = 0.152 \qquad (10)$$

As regards f_b our analysis for the turbulent boundary layer shows that this quantity is proportional to f, the coefficient of resistance in the terminal segment, the factor of proportionality being a function of L_x/L_b and the relative roughness R/k, where R is the radius of the pipe and k is a roughness factor. We have deduced from various published data on concrete pipes that for concrete $k = 0.005$ ft. on the average. This really is the equivalent sand grain size for concrete. If a smooth surface is covered with sand of this size, the flow distribution will be the same as that of concrete under the condition that mean flows and diameters are equal. In corrugated metal pipe k may be identified as the corrugation depth, that is the vertical distance between the crests and troughs of the corrugations. In the corrugated metal of Iowa tests $k = 1$ inch.

Following the above explanations, the effective frictional coefficient in the developing boundary layer may be written mathematically

$$\frac{f_b}{f} = N\left(\frac{L_x}{L_b}, \frac{R}{k}\right) \tag{11}$$

This functional relationship was investigated for the surfaces that can be interpreted as sand covered surfaces and also for corrugated metal surfaces of the type used in the Iowa tests. In these corrugated metal surfaces the corrugations are nearly sinusoidal in shape with the ratio of corrugation depths to the wave length being $k/l = 0.1875$. Since the universal law of velocities for concrete and corrugated metal surfaces are not the same, the function N was determined separately for the two surfaces. The result of computations are given in Tables 1 and 2, where f_b/f values are tabulated against L_x/L_b and R/k.

TABLE 1

EFFECTIVE COEFFICIENT OF RESISTANCE OF THE TURBULENT BOUNDARY LAYER IN SAND-COVERED PIPES

($k = 0.005$ ft. for concrete)

R/k	10	18	40	100	250	500	1000
L_x/L_b	f_b/f						
0.1	1.538	1.676	1.852	1.961	2.000	2.000	2.001
0.2	1.424	1.517	1.613	1.679	1.692	1.687	1.680
0.3	1.351	1.418	1.490	1.536	1.538	1.534	1.528
0.4	1.296	1.349	1.404	1.446	1.443	1.438	1.428
0.5	1.257	1.300	1.349	1.377	1.377	1.371	1.361
0.6	1.224	1.261	1.302	1.325	1.325	1.320	1.311
0.7	1.198	1.229	1.264	1.284	1.284	1.280	1.272
0.8	1.175	1.203	1.234	1.252	1.252	1.248	1.241
0.9	1.157	1.181	1.209	1.227	1.227	1.222	1.215
1.0	1.141	1.163	1.189	1.203	1.203	1.199	1.193

TABLE 2

EFFECTIVE COEFFICIENT OF RESISTANCE OF THE TURBULENT BOUNDARY LAYER IN CORRUGATED PIPES

($k = 0.0417$ ft. $k/l = 0.1875$)

R/k	10	18	40	100	250	500	1000
L_x/L_b	f_b/f						
0.1	1.798	1.902	2.112	2.209	2.254	2.306	2.277
0.2	1.598	1.660	1.785	1.852	1.868	1.858	1.834
0.3	1.480	1.530	1.621	1.663	1.659	1.655	1.633
0.4	1.397	1.440	1.506	1.547	1.539	1.531	1.533
0.5	1.339	1.371	1.428	1.457	1.451	1.446	1.432
0.6	1.294	1.319	1.369	1.392	1.388	1.374	1.370
0.7	1.258	1.280	1.322	1.342	1.338	1.335	1.324
0.8	1.228	1.249	1.285	1.300	1.299	1.296	1.286
0.9	1.204	1.225	1.259	1.271	1.266	1.265	1.254
1.0	1.184	1.205	1.229	1.244	1.240	1.238	1.228

As regards L_b , the length necessary for the full development of turbulent boundary layer, our analysis has shown that it depends on the type of surface, the diameter of pipe and the relative roughness. The dependence of L_b/R on R/k for the two surfaces can be read from the two curves in Figure 2.

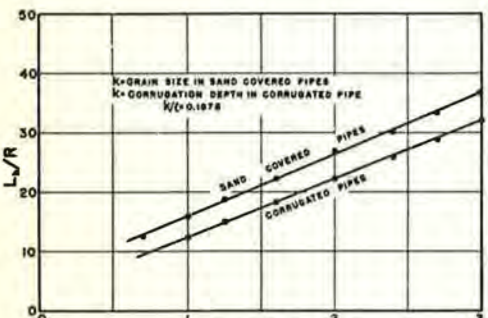

Figure 2. Length of Fully Developed Boundary Layer in Rough Pipes

Finally, for the coefficient in the terminal segment, f , we have

$$\sqrt{\frac{8}{f}} = 4.75 - 2.5\frac{k}{R} + 5.75\log(1+\frac{k}{R})^2 \cdot \log(1+\frac{R}{k}) \qquad (12)$$

for a pipe surface the hydrodynamic action of which can be simulated by a surface covered with sand of grain size k . As remarked above, for concrete k may be taken as equalling 0.005 ft. For corrugated metal of the type used in the Iowa tests

$$\sqrt{\frac{8}{f}} = 2.11 - 2.5\frac{k}{R} + 5.75\log(1+\frac{k}{R})^2 \cdot \log(1+\frac{R}{k}) \qquad (13)$$

The forms of these friction formulas differ slightly from the logarithmic forms ordinarily given in the texts on hydraulics. The difference results from the convention adopted for measuring distances in the pipe transverse sections. We measure wall distances from the top of the sand asperities or from the crests of corrugations. Accordingly, the inner diameters are measured between the crests of the roughness asperities.

IOWA EXPERIMENTS

The discharges observed in the Iowa tests for the corrugated metal and concrete pipes with square-cornered entrances are represented by small circles in Figures 3 and 4. The curves shown in the figures are theoretically obtained, using the method of computation explained above. The agreement between theory and experiment may be considered as satisfactory.

Due to the satisfactory agreement between theory and observation noted for the discharges obtained in the small size culverts of the Iowa tests, like computations were also made for eight-foot diameter corrugated metal and concrete pipe and of varying lengths up to three hundred feet. The results of the computations were compared with the similar results of the Iowa discharge formulas. It was noted that the Iowa formulas gave smaller values than the theoretical results and that the disparity in the two results increased with increasing lengths of pipe. Furthermore, the increase was larger for the corrugated metal pipe than for the concrete.

BASIS OF ANALYSIS

The analysis undertaken to determine the quantities of the boundary layer proved to be very lengthy. The details of the computations will not be given here. For the purpose of orientation, however, it may be helpful to make a few remarks about velocity distributions in general.

The momentum law, together with

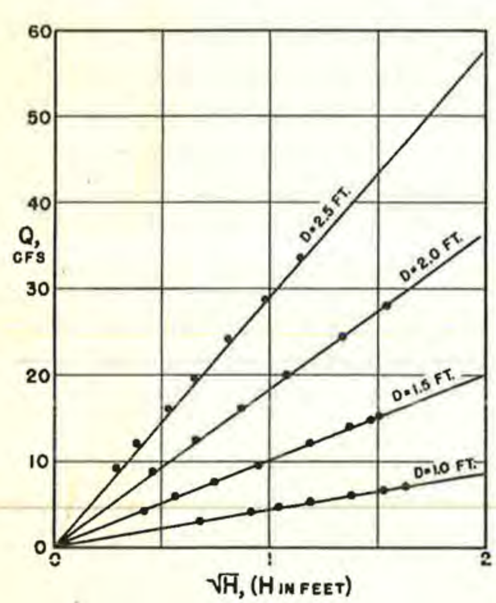

Figure 3. Iowa Test Discharges in Concrete Pipes with Square-cornered Entrances

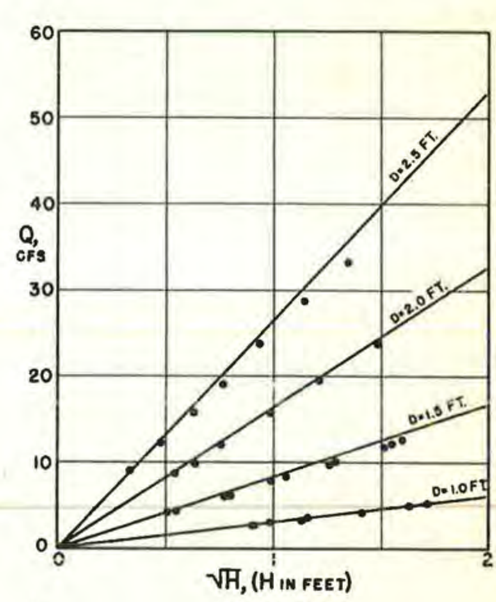

Figure 4. Iowa Test Discharges in Corrugated Pipes with Square-cornered Entrances

the supposition of a non-dissipative central core and the condition of continuity enables one to determine the variation of boundary layer thickness and wall shear with distance for a given relative roughness R/k. The computations are carried out by supposing that for each type of surface a universal velocity law exists of the type

$$\frac{u}{U_v} = a + 5.75 \log(1 + \frac{y}{k}) \qquad (14)$$

where u is the velocity at the point y, y being measured from the crests of corrugations and the tops of the hypothetical asperities of sand. The quantity U_v is the so-called shear velocity given by the relation

$$U_v = \sqrt{\frac{\tau_o}{\rho}} \qquad (15)$$

where τ_o is the shear at the wall and ρ is the density of the fluid. The quantity a is a numerical constant characteristic of a given surface. The formula is very simple and states that the velocity at a point depends solely on the shear at the wall and the distance from the wall. The extent and the limits of the fluid have no relation whatever to the velocities. Thus, the velocity law is independent of the dimensions of the central core or of the size of the pipe.

In our computations we have assumed that for a concrete surface we may take a = 5.85 and k = .005 ft. These suppositions put the concrete surface in the same class as the sand covered pipe surfaces of the Nikuradze experiments. That is, for concrete

$$\frac{u}{U_v} = 5.05 + 5.75 \log(1 + \frac{y}{k}) \qquad (16)$$

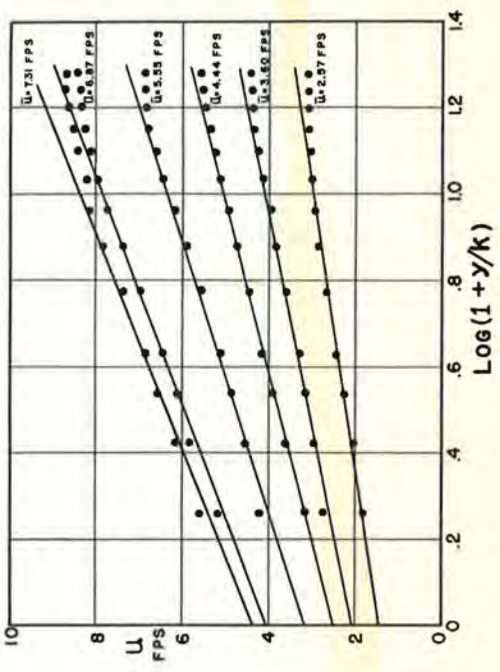

Figure 5. Velocity Distribution in Corrugated Metal Pipe; Iowa Tests

The law of velocities for a corrugated surface does not appear to have been known previously. A determination was made using a series of velocity distribution from the Iowa tests. The method of analysis in short is as follows. For any type of surface we may write

$$u = A + B \log(1 + \frac{y}{k})$$

where $A = aU_v$

and $B = 5.75\, U_v$

Thus, if u is plotted against (1 + y/k) and the plot gives a straight line, the intersection point A at the ordinate axis and the inclination B determine U_v and a. The value u/U_v may now be formed and may be plotted against log (1 + y/k). For the corrugated metal pipes the application of this method gives first the data of Figure 5 and Figure 6. According to the latter a = 8.5, and thus the law of velocities for the corrugated metal pipe is

$$\frac{u}{U_v} = 8.5 + 5.75 \log(1 + \frac{y}{k}) \qquad (17)$$

where k is corrugation depth and $k/_1$ = 0.1875.

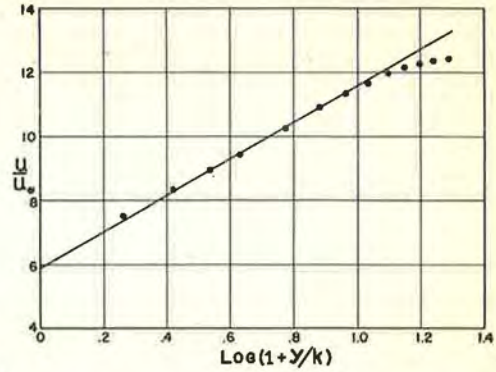

Figure 6. Universal Law of Velocities for Corrugated Metal Pipe; $k/_1$ = 0.1875