Model for Funds Allocation for Urban Highway Systems Capacity Improvements

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•MOST AGENCIES, in considering the allocation of funds for highway and street improvements among possible alternatives, make somewhat arbitrary decisions based on sufficiency ratings, engineering judgment, or some other empirical criterion. These methods for allocating funds do not consider the effect of particular improvements on the system as a whole, nor do they consider the effect on the total system costs of congestion on particular links. The objective of this paper is to introduce a feasible method of allocating funds for highway improvements in a manner that will yield a minimum total cost for the entire system, considering the total costs of operating vehicles (time, accident, and operating costs) on all links of the network, plus the total costs for making improvements to various links throughout the system.

This report is divided into three sections: use of the model, formulation of the model, and some related models. An Appendix presents the mathematical development of an efficient solution technique for the model developed, together with a solution of an example problem.

USE OF THE MODEL

Figure 1 is a flow diagram of the transportation planning in process, similar to that described by Hansen (1), illustrating how the proposed model could be used in this process. As this diagram suggests, the model performs the equivalent functions of current assignment techniques including adjustment of link travel times as volumes approach link capacities and the evaluation and adjustment of transportation network improvement plans.

The basic input data are the same as those typically required for current techniques:

1. Estimates of future zone-to-zone trip interchanges.
2. Geometry of the existing network of major arterials and express highways (to represent the network as a system of nodes and links).
3. General physical description of all existing links of the system in sufficient detail to determine practical and possible capacities and all operating costs (time, fuel, accident costs, etc.) under conditions of both free-flow and congestion.
4. Location and design characteristics of all alternative new facilities in sufficient detail to estimate items 2 and 3 for these new links or links to be considered for improvement to higher standards.
5. Cost estimates of all alternative improvements to be tested.

The precise form in which these data are required will become clear in the next section where the model is developed.

A single run of the model on a computer will simultaneously (a) assign zone-to-zone movements to the highway network according to minimum cost (largely, time costs) paths; (b) adjust costs upward as volumes exceed practical capacity; (c) adjust minimum

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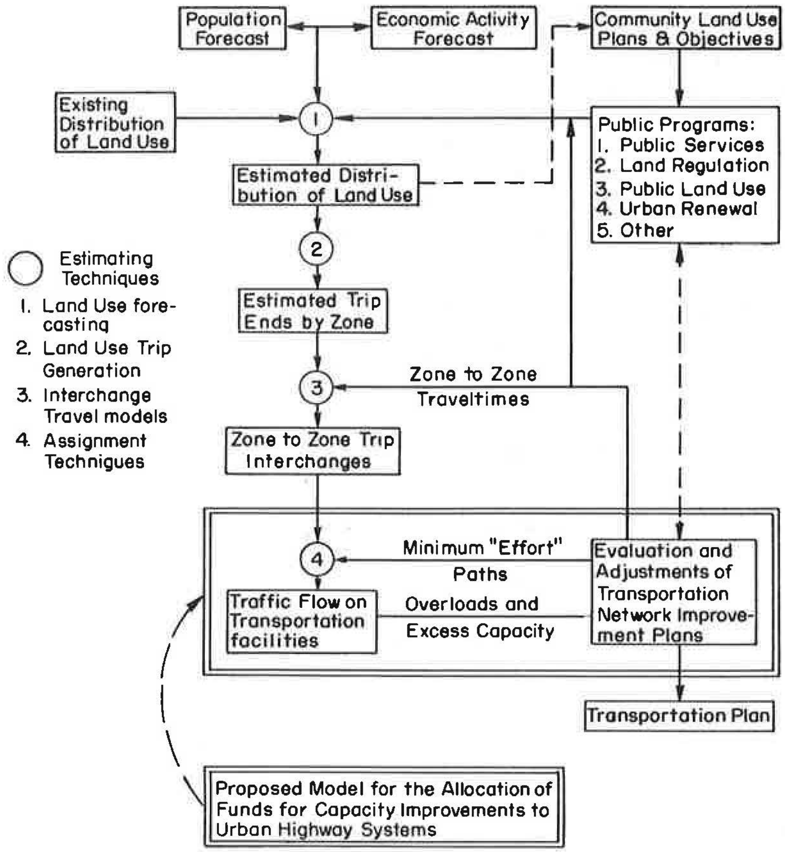

Figure 1. Transportation planning process (double-outlined boxes contain parts of process replaced by model).

cost paths as link costs are adjusted, and alter assignments accordingly; and (d) introduce new links to the system and/or increase the capacity of existing links in a manner that yields the minimum total cost system. This last operation, the addition of new capacities to the system, is done in the model by selecting, from predetermined improvements to be evaluated, those improvements that are most economical--considering the total circulation pattern and the effect that the new improvements will have on this pattern. The amount of new construction that the model will select can be limited by a budget constraint if it is desired.

Output of the model includes (a) traffic flows on all existing and proposed links, (b) capacity increases proposed for existing links and capacities of proposed new links, (c) total funds required for all new construction, (d) vehicle-operating costs for each link and for the total system, taking into account the level of congestion on each link, (e) the marginal costs for the entire system of decisions made (before operation of the model) not to allow capacity improvements to links that are congested, and (f) the marginal cost of budget limitations; i. e., the rate of return that could be realized, considering total system costs, if additional funds were available for construction. The marginal cost of budget limitations would be zero, of course, if operation of the model revealed that the budget was sufficient or even excessive. In the latter case, the model would not allocate all available funds for construction, but would indicate the most economical level of expenditures.

This model is adaptable for use in short-range planning or for small urban areas without comprehensive transportation planning studies, as well as for use in the major

metropolitan studies. On a small scale, the model could be used to program the widening of arterials and the installation of traffic control devices, or to determine such items as the system cost of decisions not to widen arterials in residential areas.

There are manifold advantages to the use of this model that cannot be realized with other current techniques, such as the following. The model can be used to determine the optimum level of public expenditures for the urban highway system. It can indicate whether budgeted funds are insufficient or excessive, and under each condition, how much money should be allocated to various improvements to yield the most economical system considering all measurable transportation costs. Also, the model can be used to determine the total system cost of decisions not to improve certain links. This may aid in the objective evaluation of controversial street widening in residential areas, and other problems of this nature. Current methods such as the benefit-cost ratio analysis only consider the effects of an improvement on one link of the system.

The assignment function of this model contains an advantage over all currently used assignment techniques which is significant enough in itself to warrant the adoption of the model. Current methods either (a) do not adjust link travel times as the volume-capacity ratio increases; or (b) adjust travel times after an assignment has been made (it is necessary then to rerun the assignment program using the new travel times, and if sufficient accuracy is desired, to continue this adjustment and reassignment until a balance is achieved); or (c) simultaneously make assignments and adjust travel times as volumes on links increase; however, assignments from particular nodes are never altered to agree with adjusted travel times after they are first assigned to the network—thus, only the assignments from the nodes considered last can be expected to be in agreement with the final adjusted travel times. In contrast, this linear programing method, in one run on the computer, assigns traffic and adjusts travel times in such a way that the final output is an assignment that is in complete balance with adjusted travel times.

There are many additional advantages that can be expected from a model that assigns traffic and programs funds for improvements all in one operation. Time and cost savings will be realized. The model could be easily used to compare alternative types of expressway systems; e.g., radial-circumferential vs grid networks. Also, it should prove interesting to compare the results that would be obtained from short-range planning in several increments vs the results from one long-range planning period. Such an analysis could indicate the optimum length of the planning period.

The use of this model by highway planning agencies is, at present, dependent not only on the availability of computers but also on the availability of personnel qualified to program models of this type on computers. Programs are widely available for solving linear programing problems using the simplex method and its variations. Small networks could be solved using these programs, but a more efficient program is needed to handle large networks on most available computers. The authors have used a solution technique developed by Charnes (2) which is very efficient for linear programing problems that have the peculiar mathematical structure of this model. This technique is amenable to computer programing; therefore, with the hope of expediting the use of this model, the authors have included the mathematical development of the method and a detailed solution of an example problem in the Appendix.

FORMULATION OF THE MODEL

In defining the objective of transportation planning, the Chicago Area Transportation Study (3, Ch. 2) listed six criteria to be strived for in the development of plans:

1. Greater speed.
2. Increased safety.
3. Lower operating costs.
4. Economy in new construction.
5. Minimizing disruption.
6. Promoting better land development.

The report discusses these criteria and contains an excellent statement of the goals of transportation planning and their relationships to comprehensive community planning goals.

Because these criteria cannot all be optimized (some conflict with others), it becomes desirable to express these goals in terms of a single criterion. The single objective chosen by CATS (3, p. 15) was "to provide that transportation system for the region which will cost least to build and use"; that is,

> to plan a system the sum of whose measurable costs for all travelers and taxpayers in the region will be at a minimum. Ideally, every cost should be included, and cost should be used in a general way to cover many indeterminate and non-measurable elements. But this form of universal social accounting is not presently possible—every cost cannot be measured. Therefore, total costs are defined here as construction and travel costs, the latter including time, accident, and other user costs.

Using the preceding as the best single criterion, it is not difficult to translate items 1 through 4 into common economic terms and compare proposed plans. The last two items are more difficult to measure, and hence cannot readily be optimized in any planning model. Careful analysis of any proposed new construction (the inputs to the model discussed here) is necessary to determine how well these last two criteria are met. However, to the extent that the effects of proposed improvements can be quantified with respect to these criteria, they should be added to construction costs and included in the inputs to the model. Otherwise, they must be evaluated with the best planning judgment. Only highway improvements that foster the desired land use planning goals and cause a minimum of disruption to the community should be considered as possible alternative improvements and therefore as inputs to the model. Hence, close cooperation between transportation and city planners is necessary in the determination of inputs to the model so that the final transportation plan will meet the last two criteria listed.

The type of planning model suggested by the preceding statement of the transportation planning objective is an optimization model whose objective function takes the following general form:

Minimize: User costs + Construction costs

in which both terms are defined in the broadest possible way. If such a model is to be readily solvable by machine methods, it should be reducible to linear programing form; i.e., it should have a linear objective function to be minimized, subject to the appropriate linear constraints on the solution.

Several difficulties arose in writing the objective function and the constraints in linear form. These difficulties and their resolution are discussed before the formal statement of the linear programing problem is made.

User costs per vehicle for a link of a given length are not constant but increase as volume on the link increases, particularly when the volume exceeds practical capacity. Therefore, the model would give an inaccurate solution if the objective function stated user costs on the links as a linear function of volume on the links. User costs per vehicle are approximately constant under free-flowing operation (volumes up to practical capacity), but increase rapidly with increasing congestion (volumes between practical and possible capacity). A good approximation of this relationship can be made using a piecewise linear function (Fig. 2)—one constant user cost per vehicle associated with free-flow conditions and another higher user cost per vehicle assigned to all vehicles that increase the volume beyond practical capacity. In the model this is accomplished by using two links (called branches of a link, to avoid confusion) to represent each link of the real network. One branch is assigned free-flow user costs and a capacity equal to practical capacity of the real link. The second branch is assigned much higher user costs (chosen so as to best fit the true relationship of average user cost per vehicle for all vehicles on the link vs volume under conditions of congestion for the real link) and a capacity equal to the difference between possible and practical capacity of the real link.

Using this technique to approximate the nonlinear relationship between volume and

user costs, the average user cost per vehicle at any level of congestion is the sum of the user costs on the two branches divided by the total number of vehicles on the two branches. At first glance it may seem that this technique is unrealistic because, when volume exceeds practical capacity, vehicles will have different user costs attached to them depending on which branch they happen to be assigned to. However, this presents no difficulty, because the model yields the optimum solution considering the total user costs on all links.

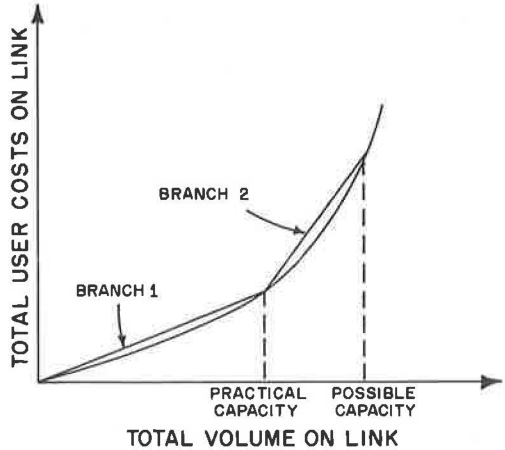

Figure 2. Piecewise linear approximation of volume vs user costs curve.

At optimum (i.e., under the final assignment), no vehicle could find a lesser cost path, except, if it were initially assigned to a high-cost branch, by exchanging branches with a vehicle on the low-cost branch of the same link. (It may be possible for a particular vehicle to displace another vehicle on the low-cost branch of a different link that is operating beyond practical capacity, and thus find a seemingly lower cost path. However, the model recognizes that such a solution would increase the cost to the displaced vehicle at least as much as the amount saved by the vehicle that displaced it and that, therefore, the overall solution is not improved. In addition, the model recognizes that neither vehicle will have actually lowered its cost in terms of average user costs per vehicle on the links. All the preceding claims of the model can be verified by mathematical proof or by the use of simple examples.) If this is done, the total system cost, the average and total link user cost, and the link assignment pattern are all unchanged. Thus, it is not of interest to know to which branch of a link a vehicle is assigned, and the only meaningful cost in the final solution is in terms of the average user cost on the links.

Thus, the model assigns traffic to the low cost branch until practical capacity of the link is reached. If the optimum solution, considering entire system costs, indicates further use of this link, the model assigns the additional traffic to the high-cost branch until possible capacity is reached. At this point, no more traffic can use the link unless it is being considered for possible improvement to higher standards (determined before operation of the model), and unless total system costs indicate that is the most economical decision to be made.

Treating an improvement to a link of the system (or the addition of a new link) as a capacity increase allows some of the vehicles to travel at a lower cost (if such is the case) and also increases the maximum number of vehicles that may use the link. By viewing improvements in this manner, the model adds the unit cost of operating a vehicle to the unit cost of providing the additional capacity to handle another vehicle.

If the total costs of an improvement are converted to daily costs over the useful life of the improvement, and then divided by the added daily capacity, the result is the cost of providing the additional capacity for one vehicle trip over the link. The cost of providing additional capacity for vehicles to use a link does not increase linearly as the amount of capacity added increases. The cost of highway and street improvements takes the form of step functions. That is, a street-widening project might add a given volume to the capacity of the original facility, the installation of a signal would add perhaps a different volume, and so on, depending on the specific type of improvement. For this reason, the unit cost of adding capacity to a particular link would not be constant but would vary with the type of improvement contemplated (Fig. 3).

The use of a constant unit cost for increasing the capacity of links or adding new links will result in values that show the relative demand (from a cost viewpoint) for improvement on the various links. Thus, a solution that indicated the addition of a very small number of vehicles to the capacity of a particular arterial would not be

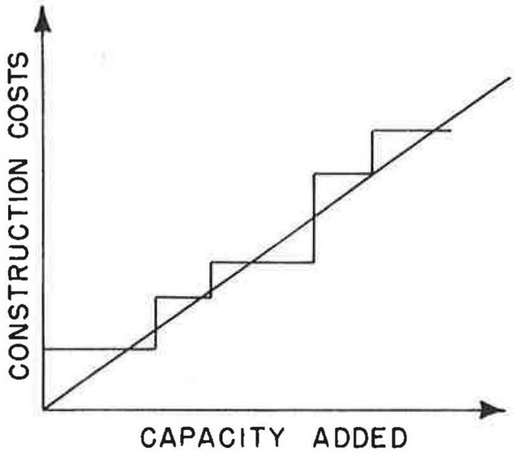

Figure 3. Linear approximation of unit cost of adding capacity.

significant, whereas the solution that indicated the addition of several hundred vehicles to the capacity of a link definitely points out a critical link—one whose improvement will contribute greatly to the efficiency of the entire system.

The preceding discussion indicates the form that the objective function must take. Also indicated is the nature of one of the necessary sets of constraints (the capacity restrictions) and the nature of one optional constraint (the budget limitation on the amount of funds that may be allocated for improvements to the network). Two additional sets of constraints are necessary due to the structure of the model.

Due to the manner in which the problem of the nonlinearity of user costs was handled (the representation of each link by one branch for free-flow conditions and one branch for congested-flow conditions), it is necessary to introduce a constraint that insures that, when the capacity of a link is increased, the ratio of practical to possible capacity will remain the same. This follows from the definitions of practical and possible capacities. If this constraint was not included, the model could, for example, double the practical capacity of a link by carrying out an indicated improvement without adding any incremental capacity (possible minus practical capacity). Therefore, another set of constraints is required, one for each link to be considered for improvement.

The final set of constraints required, aside from the non-negativity conditions on all the variables, is the Kirchhoff node equations, so called because of the analogy to electrical network equations. These state that there must be a balance of flow at each node; i.e., the flow into a node must equal the flow out of the node, considering traffic originating at a node as input and traffic with destination at a node as output. Due to the structure of the model, one of these equations is required for each node for each copy (a copy is the distribution from a node of origin). For example, if a network has ten nodes and five nodes with traffic originating from them, then $10 \times 5 = 50$ Kirchhoff node equations are required.

The formal statement of the preceding discussion takes the following form as a linear programing problem:

$$\text{Min } f(X, C') = \sum_i \sum_j \sum_k U_{jk} (X_{ijk+} + X_{ijk-}) + \sum_j r_j C'_{j1}$$

subject to

$$\sum_j \sum_k e_{hijk} (X_{ijk+} - X_{ijk-}) = E_{hi} \qquad h = 1, \ldots, n; \quad i = 1, \ldots, m$$

$$\sum_i (X_{ijk+} + X_{ijk-}) - C'_{jk} \le C_{jk} \qquad j = 1, \ldots, L; \quad k = 1, 2$$

$$\sum_j r_j C'_{j1} \le F$$

$$C_{j1} C'_{j2} - C_{j2} C'_{j1} = 0 \qquad J = 1, \ldots, L$$

$$X_{ijk+}, X_{ijk-}, C'_{jk} \ge 0$$

in which

X_{ijk+} = number of vehicles per day on k^{th} branch of j^{th} link for distribution from the i^{th} originating node in arbitrarily chosen positive direction of branch. X_{ijk-} is volume in opposite direction.

U_{jk} = user cost per vehicle on k^{th} branch, j^{th} link.

r_j = cost of improvement per day per vehicle of capacity added to j^{th} link.

C_{jk} = existing two-way daily capacity of k^{th} branch, j^{th} link.

C'_{jk} = daily capacity added to k^{th} branch, j^{th} link.

e_{hijk} = incidence number at h^{th} node, for flow from i^{th} originating node, for k^{th} branch of j^{th} link. Convention adopted here for incidence numbers is

+ 1 for input to node; i.e., if arbitrarily chosen positive direction of a link is toward node.
- 1 for output from node.
0 if link is not connected to node.

E_{hi} = number of vehicles per day originating (minus) or terminating (plus) at h^{th} node for flow from i^{th} originating node.

F = total funds available for improvements to the network on a daily basis.

The first set of constraints is the Kirchhoff node equation stating that the total traffic flow into each node must equal the total flow out.

The second set of constraints states that the total traffic minus the added capacity cannot exceed the existing capacity of each branch.

The third constraint states that the total amount spent for capacity improvements cannot exceed the amount of funds available.

The fourth set of constraints states that the improvements in capacity must be made so that the ratio of practical to possible capacity remains the same for each link.

The dual of this problem can be written as

$$\text{Max } f(B, D, G) = \sum_h \sum_i B_{hi} E_{hi} - FD - \sum_j \sum_k C_{jk} G_{jk}$$

subject to

$$(\text{sign of } X_{ijk}) \sum_h e_{hijk} B_{hi} - G_{jk} \le U_{jk} \qquad i = 1, \ldots, m; \; j = 1, \ldots, L; \; k = 1, 2$$

$$G_{j1} - C_{j2} D_j - r_j D \le r_j \qquad j = 1, \ldots, L$$

$$G_{j2} + C_{j1} D_j \le 0 \qquad j = 1, \ldots, L$$

$$G_{jk}, \; D \le 0$$

B_{hi}, D_j may be positive, negative, or zero

in which

E_{hi}, F, C_{jk}, X_{ijk+}, X_{ijk-}, e_{hijk}, U_{jk} and r_j are as defined for the primal problem.

The interpretation of the dual appears to be of little utility; however, two of the dual variables, whose values will be known when the primal problem is solved, are of considerable interest:

G_{jk} = total change in system cost per unit increase in capacity of k^{th} branch, j^{th} link.

D = total change in system cost per unit increase in F, budget limitation.

As already noted these values will be either negative or zero, and therefore will indicate the unit savings that would have been realized, if capacities could be increased on links not considered for improvement or if additional funds were available for improvements.

The structure of the general problem and the relationship between the primal and the dual are given in Table 1.

TABLE 1

GENERAL FORM OF PRIMAL AND DUAL TABLEAU FOR CAPACITY IMPROVEMENT MODEL

Interpretation of Primal Constraint		$X_{111+}\ X_{111-} \cdots X_{1L2+}\ X_{1L2-}$	$\cdots$	$X_{m11+}\ X_{m11-} \cdots X_{mL2+}\ X_{mL2-}$	$C'_{11}\ C'_{12} \cdots C'_{L1}\ C'_{L2}$	
Balance of Input and Output of Traffic at each Node	B_{11}	$e_{1111}\ -e_{1111} \cdots e_{11L2}\ -e_{11L2}$				$= E_{11}$
	B_{12}	$e_{1211}\ -e_{1211} \cdots e_{12L2}\ -e_{12L2}$				$= E_{12}$
	$\vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$				$\vdots$
	B_{1n}	$e_{1n11}\ -e_{1n11}\ \ e_{1nL2}\ -e_{1nL2}$				$= E_{1n}$
	$\vdots$		$\ddots$			$\vdots$
	B_{m1}			$e_{m111}\ -e_{m111} \cdots e_{m1L2}\ -e_{m1L2}$		$= E_{m1}$
	B_{m}			$e_{m211}\ -e_{m211} \cdots e_{m2L2}\ -e_{m2L2}$		$= E_{m2}$
	$\vdots$			$\vdots \quad \vdots \quad \vdots \quad \vdots$		$\vdots$
	B_{mn}			$e_{mn11}\ -e_{mn11} \cdots e_{mnL2}\ -e_{mnL2}$		$= E_{mn}$
Funds Constraint	D				$-r_1 \cdots -r_L$	$\geq -F$
Rentention of Capacity Ratio (Pass.: Pract.) for each Link	D_1				$-C_{12}\ C_{11}$	$= 0$
	$\vdots$				$\ddots$	
	D_L				$-C_{L2}\ C_{L1}$	$= 0$
Capacity Constraint for each Branch of each Link	G_{11}	$-1\ \ -1$	$\cdots$	$-1\ \ -1$	$+1$	$\geq -C_{11}$
	$\vdots$	$\ddots$	$\cdots$	$\ddots$	$\ddots$	$\vdots$
	G_{L2}	$-1\ \ -1$	$\cdots$	$-1\ \ -1$	$+1$	$\geq -C_{L2}$
		$\leq U_{11}\ \ \leq U_{11} \cdots \leq U_{L2}\ \ \leq U_{L2}$	$\cdots$	$\leq U_{11}\ \ \leq U_{11} \cdots \leq U_{L2}\ \ \leq U_{L2}$	$\leq r_1\ \ \leq 0 \cdots \leq r_L\ \ \leq 0$	

SOME RELATED MODELS

At least three other writers have suggested the use of linear programing models to determine the optimum allocation of funds for highway improvements. Each author has considered different aspects of the general problem, and therefore has attacked the problem in a different manner. The first two of these models are mentioned only briefly, because they have already appeared in the literature. The work of the third author, however, is contained in an unpublished thesis; therefore, it should be helpful to discuss this model in more detail.

Garrison (4) considered the problem of the shipment of commodities between urban centers over a regional or National highway network. Assumed given are existing capacities between all pairs of cities, shipment costs on all links, current levels of production and demand for commodities at all nodes (cities), unit costs of adding capacities to all links, and quantitative measures of the impact of capacity improvements on production and demand at all nodes. Solution of the model determines the optimum allocation of funds for improvements to the highway system. The model also indicates the total growth in production and demand that is expected to occur in each city as a result of the added capacities of the links. Unlike the model presented in this paper, Garrison's model does not, as written, handle such difficulties as the nonlinearity of shipment costs, the fact that commodities are transshipped through several nodes in the real world, or the fact that the solution may be constrained by budget limitations. However, these difficulties could be dealt with in the same manner as in the urban model presented in this paper.

Quandt (5) also considered the problem of the shipment of commodities between centers of production and consumption and the allocation of funds for highway improvements. Several linear programing formulations are presented, each dealing with the problem under different assumptions. In general, this work differs from Garrison's in that transshipment through several cities is possible, and the case where budget constraints are introduced is considered. However, Quandt does not deal with increases in production or demand, nor with the problem of nonlinearity of shipment costs versus volume on links.

Plaza (6) also deals with interurban travel and the allocation of funds for highway improvements. However, he is not concerned with the determination of commodity flows between cities. As in the preceding urban model, node-to-node traffic is assumed known. In fact, the actual volumes on links are assumed to have been determined by an assignment model. Unlike the urban model, however, Plaza's formulation ignores capacity restrictions and concentrates on the determination of the optimum allocation of funds to improve links to higher standards in order to reduce maintenance costs and user costs. This model has application to interurban highways where traffic volumes never exceed practical capacity but where geometric design features and roadway surface conditions are the chief factors affecting user costs. The model has particular utility in the determination of the timing of improvements to higher-type surfaces in developing regions.

It is not possible to discuss here all the many variations of this problem that were treated by Plaza. Only the linear programing formulation covering the most general case of the problem is presented.

The general problem is

$$\text{Min} \quad \sum_i t_i^q \left[K_i - d_i^q \sum_k x_i^k \right]$$

subject to

$$\sum_i \left[\sum_{k=1}^{k} r_i^k x_i^k + a_i^k x_i^k \right] + B_o^k \leq B^k$$

$$\sum_{k=1}^{q} x_i^k \leq G_i$$

$$x_i^k \geq 0$$

in which

t_i^q = total number of vehicles using i^{th} link in q^{th} year.
q = number of years to accomplish program.
K_i = $u_i^q S_i + U_i^q G_i$ = cost of travel over i^{th} link in q^{th} year, if no improvement were made.
u_i^q = time plus operating costs during q^{th} year of travel over unimproved sections of i^{th} link, per vehicle mile.
S_i = length of previously improved sections in i^{th} link, in miles.
G_i = length of unimproved sections in i^{th} link, in miles.
d_i^q = difference of travel costs per mile during q^{th} year over unimproved and improved sections of i^{th} link. $d_i^q = U_i^q - u_i^q$
x_i^k = length of i^{th} link improved during k^{th} year of program (these are variables to be determined by solution of problem).
r_i^k = $b_i^k - c_i^k$ = difference in costs of maintenance of a miles of i^{th} link after and before improvement at k^{th} year.
a_i^k = cost of improvement per mile of i^{th} link, k^{th} year.
B^k = budget for maintenance and improvements on whole network during k^{th} year.

$$B_0^k = \sum_i (b_i^k S_i + c_i^k G_i) = \text{budget at } k^{th} \text{ year, if no improvements at all are made.}$$

The preceding objective function is the total travel costs over the network after the improvements have been made.

The first set of constraints states that the sum of all increases in maintenance costs due to improvements plus the sum of all costs of improvements cannot exceed the budget in any year.

The second set of constraints states that the total length of improvements over the years, to any link, cannot exceed the original length of unimproved section on that link.

The following are some of the many particular cases of this problem considered and successfully solved by Plaza:

1. The budget limitation only applies to the cost of improvements.
2. Cost of improvements per mile for each section is expected to remain constant throughout the period of the program.
3. Budget constraints are unspecified; the optimum level of expenditure for each year is to be determined.
4. Scarcity of materials required for certain improvements.
5. Geographic distribution of the improvements is desired to avoid excessive concentration of improvements in particular areas.
6. Decentralization of the budget among several local agencies.
7. Changes in the pattern of movements between cities due to improvements in the system.

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Appendix

DEVELOPMENT OF A COMPUTATIONAL TECHNIQUE

The linear programing problem in the second section requires nm + 3L + 1 constraint equations, in which n is the number of nodes in the network, m is the number of copies (nodes that originate traffic), and L is the number of links. The analysis of relatively large street systems by simplex routines would exceed the capacity of most electronic computers. Therefore, a computational technique that takes advantage of the special structure of the proposed model would be very useful. The structure of the model is shown in Figure 4.

The M_i's (matrices of incidence numbers, e_{hijk}'s for the i^{th} copy) together with their respective restraint vectors b_i's) represent individual linear programing problems of which the first m are unconstrained network problems for the distribution of traffic from each copy. The capacity improvement copy M_{m+1} b_{m+1} represents the linear programing problem for the capacity ratio and fund constraints. The N_i's are matrices of structural coefficients (1 or 0) that couple the various copies to the link capacity constraints (C's). Y_i is the solution vector of flows (or capacity increments for i = m+1)

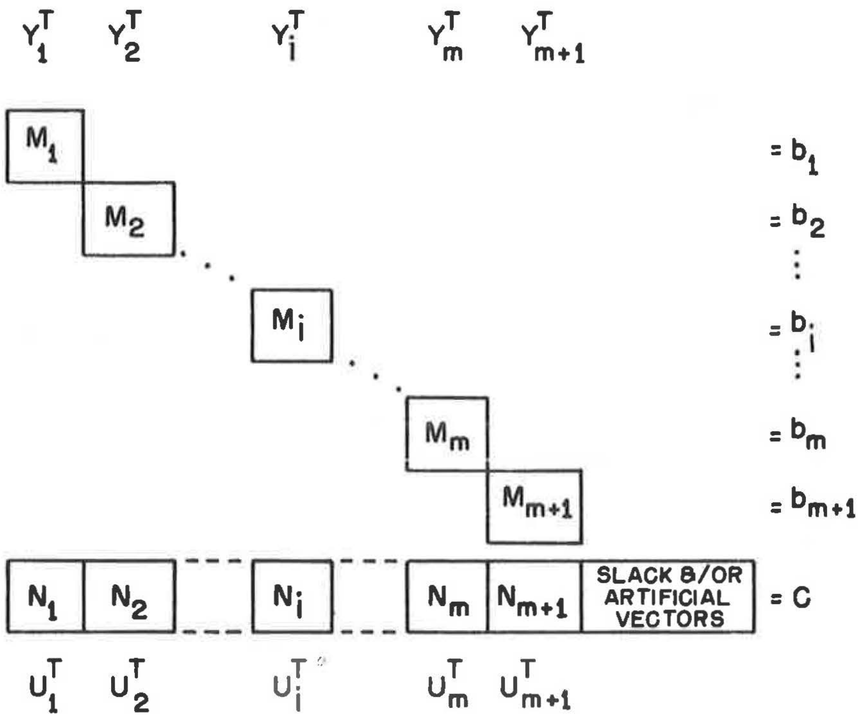

Figure 4. Model schema.

for the i^{th} copy, and b_i is the vector of stipulations for the i^{th} copy.

The form of this problem is the same as that of the mixing routine for coupling models, developed by Charnes (2). This method essentially solves two different linear programing problems: the network problem for each copy and the mixing problem. The mixing problem uses a convex combination (mixture) of extreme point solutions from each copy to obtain (by iteration) the optimum solution for the entire system.

The mixing problem is stated as follows:

$$\text{Minimize} \sum_i \sum_q U_i^T p_{iq} R_{iq}$$

subject to

$$\sum_i \sum_q N_i p_{iq} R_{iq} = C$$

$$\sum_q R_{iq} = 1$$

$$R_{iq} \geq 0$$

in which

U_i = vector of costs (U_{jk}) for i^{th} copy.

R_{iq} = fraction of q^{th} extreme point (p_{iq}) from i^{th} copy that is included in mixing problem.

p_{iq} = q^{th} extreme point (solution vector) from i^{th} copy.

U_{jk}, N_i, and C are as previously defined.

The vector of coefficients in the mixing problem associated with R_{iq} is

$$P_{iq} = \left[H_{iq}^T \mid d_i^T \right]$$

in which

$H_{iq} = N_i \, p_{iq}$

d_i^T = unit vector with a 1 in i^{th} column.

$U_{iq} = U_i^T p_{iq}$ = solution cost for iq^{th} extreme point.

Because each network copy has only n-1 constraint equations, the capacity improvement copy has only L+1 equations and the mixing problem has only m+2L equations (even with all branches constrained, which is never the case for an actual street system), the mixing routine enables quite large networks to be programed for electronic computer solution.

EXAMPLE

The problem chosen to illustrate this method is shown in Figure 5 and Table 2. It is a somewhat simplified (for the purpose of hand computations) representation of an existing city street network. The link operating costs (cents per vehicle) are equivalent to the average vehicle operating costs per mile times the length of the link plus a time cost of traversing the link and an estimate of the accident costs on the link. The practical and possible capacities of the streets were determined from a physical description of the width, control of access, adjacent land use, and other pertinent data.

The capacity figures used are 24-hr, two-directional ones. If the relationship between peak-hour and 24-hr volumes is kept in mind, either peak-hour or 24-hr volumes may be used.

The node origins and destinations (Table 2) are based on data from an actual traffic origin and destination study. Traffic was assigned from the originating node to the various destinations for each copy by the uncapacitated network minimum cost path method. From Table 3, which gives the total assignment from all three copies, it is evident that only two links in the system (links 5 and 9) have assignments over practical capacity. Only those links close to or over capacity are likely to be critical ones. Thus it is possible to simplify the required computations tremendously by assuming that only links 5 and 9 are capacitated and to solve the coupling problem with the remaining links as uncapacitated ones. If, however, in the optimal solution (with this assumption) other links have assignments greater than their actual practical capacity, the problem would have to be reworked—with these links now considered as capacitated ones. Because it is of interest to determine the "cost" of making the political or policy decision not to improve a particular street in a system, it was assumed that the capacity of link 9 could not be improved. The result of this decision can be evaluated by the dual variables associated with link 9 in the final solution.

To have amounts that are easier to work with by hand-computing techniques, the origin and destination amounts in the computations are in hundreds of vehicles and the costs, therefore, in dollars per 100 vehicles. The incidence matrix (M_4) for the capacity improvement copy is given in Table 4. S_F is the fund slack vector; C'_{51} and C'_{52} (as previously defined) are the daily capacities added to branches 51 and 52; U_j is the cost of adding one vehicle of daily capacity to link j (calculated as explained in the second section); and b_4 is the stipulations vector.

The $50 budget stipulation is the total daily funds for improvements to the system.

An initial solution for the mixing problem was composed of one extreme point from each of the four copies plus four slack and artificial vectors required to satisfy the capacity constraints.[1] For the three network copies, these were obtained by multiplying

[1] A slightly different solution for this type of problem is given elsewhere (7).

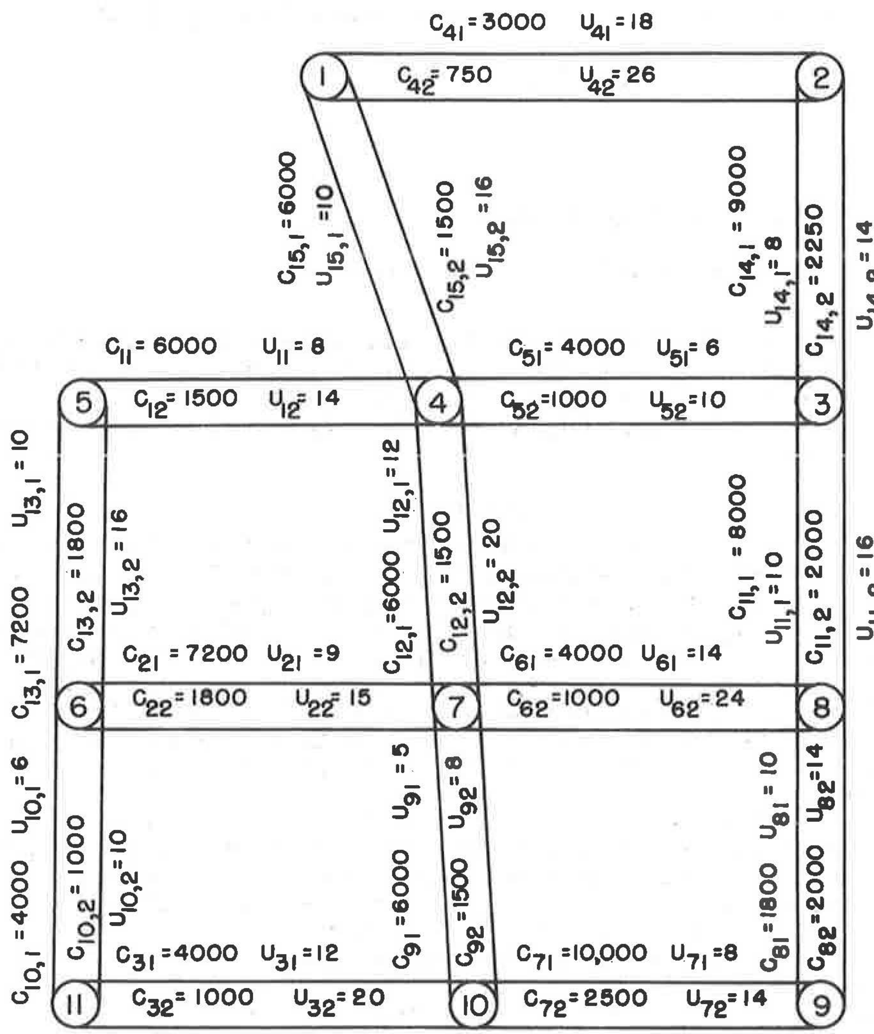

Figure 5. Operating costs per vehicle (U_{jk}) and capacity (C_{jk}) of each branch of each link for example problem.

TABLE 2

ZONE-TO-ZONE MOVEMENTS[a]

Zone (node)	Copy 1	Copy 2	Copy 3
1	500	3,000	500
2	-8,000	1,000	500
3	0	1,000	3,000
4	1,000	500	500
5	0	1,000	500
6	500	1,000	-10,000
7	1,000	2,000	2,000
8	500	0	1,000
9	2,500	-10,000	500
10	1,500	0	500
11	500	500	500

[a] Negative sign = traffic originates at node; positive number = traffic that has destination at node.

TABLE 3

CAPACITIES AND INITIAL TRAFFIC ASSIGNMENT TO NETWORK LINKS[a]

Link	Capacity of Branch 1 of j^{th} Link, C_{j1}	Capacity of Branch 2 of j^{th} Link, C_{j2}	Total Traffic Assigned[b]
1	6,000	1,500	5,500
2	7,200	1,800	6,500
3	4,000	1,000	500
4	3,000	750	500
5	4,000	1,000	8,000
6	4,000	1,000	1,000
7	10,000	2,500	8,500
8	8,000	2,000	4,500
9	6,000	1,500	10,500
10	4,000	1,000	1,000
11	8,000	2,000	5,000
12	6,000	1,500	6,000
13	7,200	1,800	7,000
14	9,000	2,250	9,000
15	6,000	1,500	3,500

[a]Positive directions of links chosen in obviously expected direction of traffic distribution for each copy, but this is arbitrary choice.

[b]From all copies assigned by uncapacitated minimum cost path solution.

TABLE 4

SIMPLEX TABLEAU (M_4) FOR CAPACITY IMPROVEMENT COPY

r_j	5	0	0	
	C'_{51}	C'_{52}	S_F	b_4
	5	0	1	50
	-10	40	0	0

the corresponding N_i (incident matrix) times the uncapacitated solution, $H_{iq} = N_i p_{iq}$, and attaching a unit vector with a 1 in the i^{th} row. Thus, $P_{11}^T = (H_{11}, 1, 0, 0, 0) = (45, 0, 15, 0, 1, 0, 0, 0)$; (Table 5).

The extreme point from the fourth (capacity improvement) copy was computed by solving M_4 (the fourth incidence matrix) using the regular simplex method.

At this point the computational procedure is outlined as an algorithm:

1. A basic feasible optimal solution is obtained from each copy, and costs associated with each solution are computed. The uncapacitated minimum cost network or the simplex method (copy 4) gives the copy solutions. Solution costs are given by

$$U_{iq} = U_i^T p_{iq}$$

2. Mixing problem vectors are obtained from the first extreme points (p_{i1}) of the individual copies by using

$$p_{i1} = (H_{i1}^T \mid d_i^T = \left[(N_i p_{i1})^T \mid d_i^T\right]$$

3. A basis (B) for the mixing problem is obtained by adding slack and artificial vectors to satisfy the stipulations.

4. The initial basis is inverted so that the modified simplex method can be used (by iteration) to obtain an optimum solution.

TABLE 5

INITIAL BASIS FOR MIXING ROUTINE AND STIPULATION VECTOR P_0

V	2095	2555	1850	0	K	0	K	0	
Basis	P_{11}	P_{21}	P_{31}	P_{41}	A_{51}	S_{52}	A_{91}	S_{92}	P_0
	45	0	35	0	-1	0	0	0	40
	0	0	0	0	0	1	0	0	10
	15	75	15	0	0	0	-1	0	60
	0	0	0	0	0	0	0	1	15
	1	0	0	0	0	0	0	0	1
	0	1	0	0	0	0	0	0	1
	0	0	1	0	0	0	0	0	1
	0	0	0	1	0	0	0	0	1

5. The P_0' column for the first tableau (inverted first basis) is calculated from

$$P_0' = B^{-1}\left[C|1,\ 1,\ 1,\ 1\right]^T$$

For this problem (Table 6), $P_0' = B^{-1}\left[C_{51},\ C_{52},\ C_{91},\ C_{92},\ 1,\ 1,\ 1,\ 1\right]^T$

6. The vector of costs for the first basis vectors is V_B, of which the first m-components are the costs of the copy solutions. (For this example the artificial vectors were assigned a finite, but very large, "K" cost and the slack vectors have a zero cost.) The w^T row is obtained from

$$w^T = V_B^T B^{-1}.$$

7. The mixing problem solution is tested for optimality by computing a new (q^{th}) optimal solution from one of the copies (in the case of the first check, copy 1 was used) using the regular network algorithm (or the simplex method in the case of copy 4), but with new "dummy" costs. The dummy costs (J) are obtained from

$$J^T = U_i^T - W^T N_i$$

in which

W^T = portion of w^T row associated with capacitated links (here, the first four elements).

J = vector of dummy link costs used to calculate a new optimum from one of the copies.

The cost of this new optimal copy solution (p_{iq}) is obtained from

$$U_{iq} = U_i^T p_{iq}$$

8. As in step 2, the new optimal copy solution is transformed into a mixing problem vector, P_{iq}.

9. Next, entry into the basis is tested for by computing (similarly to the regular simplex method)

$$Z_{iq} - U_{iq} = w^T P_{iq} - U_{iq}$$

If $Z_{iq} - U_{iq} > 0$, the vector can improve the mixing problem solution and, therefore, enters the basis (one proceeds to step 10).

If $Z_{iq} - U_{iq} \leq 0$, the vector will not improve the mixing problem solution. Therefore, step 7 is returned to and repeated with the next copy. When no vector from any copy wants to come into the mixing problem basis, the mixing routine is optimal. Therefore, for optimality,

$$\underset{q}{\text{maximum}}\ w^T P_{iq} - U_{iq} \leq 0$$

for each $i = 1, \ldots, m+1$.

10. The new mixing problem vector (from step 8) is premultiplied by the current inverted basis, B^{-1}, to yield $B^{-1}P_{iq}$.

11. The vector to be removed is determined by dividing the P_0' elements by the corresponding positive elements of $B^{-1}P_{iq}$. The minimum ratio of P_0' elements to $B^{-1}P_{iq}$ element signifies the vector to be removed.

12. By row reduction, $B^{-1}P_{iq}$ is transformed into a unit vector with a 1 in the row having the minimum ratio (from step 11). This same row reduction is also performed on the complete inverted basis, including the w^T row. The result is the complete next-stage tableau.

13. Step 7 is returned to the new optimum solution calculated from the same copy that was being analyzed in steps 10 through 12. This is continued until at step 9 an optimum is indicated.

Table 5 is the initial mixing problem basis. Table 6 gives the inverse of this basis, along with the cost vector V_B, P_{12} (calculated from copy 1, p_{iq}, using dummy costs determined as outlined in step 7), and $B^{-1}P_{12}$. This table shows that P_{12} will improve the solution ($Z_{12} - U_{12} = 60k - 195$) and that S_{52} is the vector to be removed from the basis.

The upper half of Table 7 gives the result (stage 2) of bringing P_{12} into and of removing S_{52} from the basis. Iteration continued (following the outlined algorithm) until

TABLE 6

INVERSE OF INITIAL BASIS AND FIRST ITERATION (MODIFIED SIMPLEX TABLEAU) OF MIXING ROUTINE TESTING FOR OPTIMALITY

P_{12}	V_B	BASIS									P_0'	$B^{-1}P_{12}$	RATIO
0	2095	P_{11}					1				1	1	1
30	2555	P_{21}						1			1	0	
0	1850	P_{31}							1		1	0	
0	0	P_{41}								1	1	0	
1	K	A_{51}	-1				45		35		40	45	8/9
0	0	S_{52}		1							10	30	1/3
0	K	A_{91}			-1		15	75	15		45	15	3
0	0	S_{92}				1					15	0	
U_{12} = 2290		W	-K	0	-K	0	2095 +60K	2555 +75K	1850 +50K	0	6500 +85K	-195 +60K	

TABLE 7

TABLEAUX FOR 2ND AND 16TH (FINAL) ITERATIONS

			P_{13}	V_B	B_2^{-1}									P_0'	$B^{-1}P_{13}$	RATIO		
			0	2095	P_{11}		$-1/30$			1				$2/3$	1	$2/3$		
			0	2555	P_{21}						1			1				
			0	1850	P_{31}							1		1				
			0	0	P_{41}								1	1				
			1	K	A_{51}	-1	$-3/2$			45		35		25	45	$5/9$		
			0	2290	P_{12}		$1/30$							$1/3$				
			0	K	A_{91}		$-1/2$	-1		15	75	15		40	15	$8/3$		
			0	0	S_{92}				1					15				
			2460		W	-K	$13/2$ -2K	-K	0	2095 +60K	2555 +75K	1850 +50K	0	6565 +65K	-365 -60K			
P_{44}	P_{34}	P_{27}	$P_{1,10}$	V_B	B_{16}^{-1}									P_0'	$B^{-1}P_{1,10}$	$B^{-1}P_{27}$	$B^{-1}P_{34}$	$B^{-1}P_{44}$
-10	35	0	30	2230	P_{15}	$-1/10$	$-1/10$			3		$7/2$	$-5/4$	$1/4$	0	0	0	0
$-5/2$	0	0	0	2555	P_{21}			$1/20$	$11/300$		$-11/4$	$-3/4$		$1/20$	0	1	0	0
0	15	75	0	1850	P_{31}		$-1/35$					1	$-1/14$	$-9/14$	0	0	1	0
0	0	0	0	2170	P_{13}	$1/10$	$1/10$			-2		$7/2$	$5/4$	$3/4$	1	0	0	0
0	0	0	1	50									1	1	0	0	0	1
0	0	1	0	1990	P_{32}		$1/35$						$1/4$	$5/14$	0	0	0	0
0	1	0	0	2635	P_{25}			$-1/20$	$-1/20$		$15/4$	$3/4$		$3/4$	0	0	0	0
1	0	0	0	2780	P_{22}				$1/75$					$1/5$	0	0	0	0
50	1850	2555	2170		W	-6	-2	-4	-1	2350	2855	2120	-15	6795	Z-U= 0	Z-U= 0	Z-U= 0	Z-U= 0

an optimum solution was indicated at stage 16. The lower half of Table 7 shows the 16th stage.

The elements of P_0' in this final stage indicate the proportion (R_{iq}) of the various copy solutions that are combined to yield the optimum solution for the entire system. Taking these P_0' elements times their corresponding vector of flows and summing gives the total traffic flow on all links of the system (Fig. 6) for the optimal mixing problem solution.

Although the mixing routine always yields a feasible optimal solution, the results are not always basic feasible optimal. Therefore, the regular network evaluation procedure was used to compute the node evaluations (G's and w's) for the three copy solutions and to obtain a basic feasible optimal solution (2, p. 637). The evaluation procedure usually produces a basic feasible solution with only one or two iterations from the mixing routine results. In this example, the mixing model results were basic; therefore, no iterations were necessary.

The optimal solution of this example (Fig. 6) shows that the traffic flow on link 9 is at the maximum allowable capacity (assuming that the capacity of link 9 could not be increased). Also, the practical capacity of link 5 was increased from 4,000 to 5,000 vehicles, and the difference between practical and possible capacities increased from 1,000 to 1,250 vehicles. This increase in capacity expended all available funds. Because the flows on all other links are below their practical capacities, the system does not have to be re-evaluated when the assumption that these links are uncapacitated is relaxed and they become capacitated links.

The G^*_{jk} associated with the capacity constraints on link 9, $G^*_{91} = -4$ and $G^*_{92} = -1$ (Table 7), are the evaluations of the functional for a unit change in the capacity constraints C_{91} and C_{92}. Thus, these dual evaluators give the decrease to the total system cost if a vehicle unit of capacity could be added to link 9. Similarly, $G^*_{51} = -6$ and $G^*_{52} = -2$ give the savings that would be realized in the system cost if the capacity of link 5 could be increased by one unit.

The savings to the system for an additional unit of investment in capacity increase (-D) can also be obtained from the w row. The eighth value divided by the total funds is -15/50 = 30 percent.

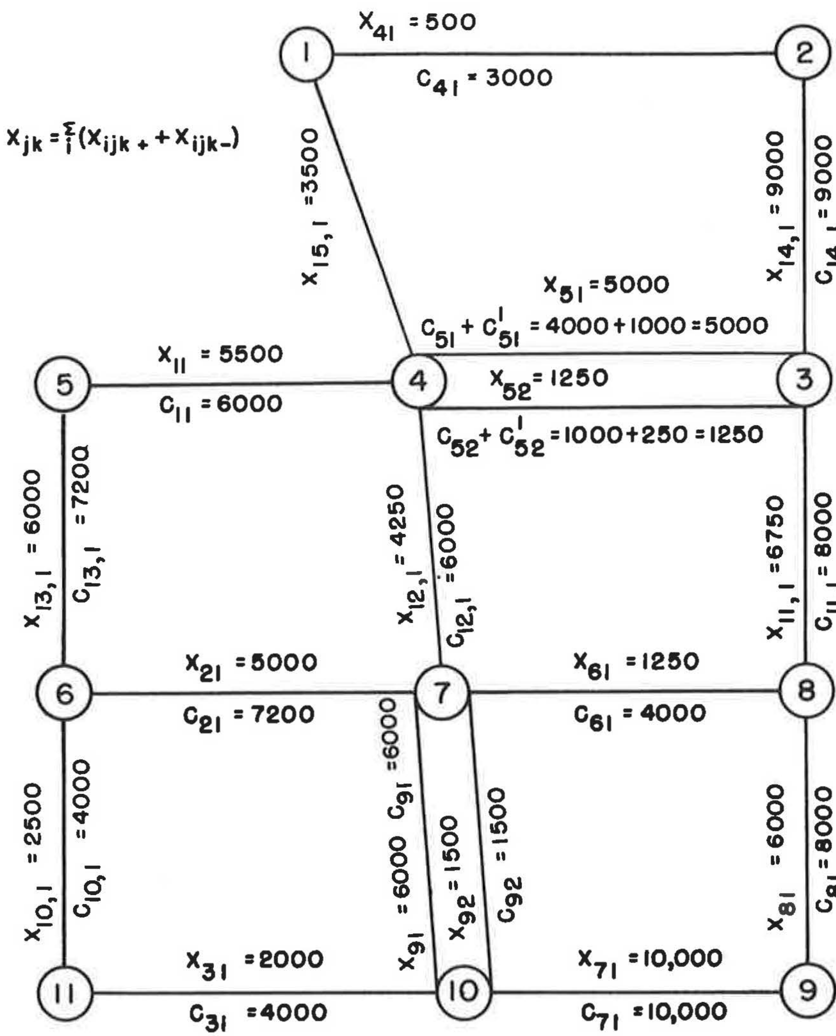

Figure 6. Optimum solution for example problem (from mixing routine) showing total traffic (X_{jk}) on links, original capacities (C_{jk}), and capacity increases (C'_{jk}).

By looking at the dual problem constraints,

$$G_{j1} - C_{j2}D_j - r_jD \le r_j$$

$$G_{j2} + C_{j1}D_j \le 0$$

$$G_{51} - C_{52}D_5 - 5D \le 5$$

$$G_{52} + C_{51}D_5 \le 0$$

Because at the optimum there was no slack, the equalities hold; therefore,

$$6 - 1{,}000\ D_5 - 5D = 5$$
$$2 + 4{,}000\ D_5 = 0;$$
$$D_5 = 2/4{,}000,$$

Substituting,

$$6 - 1{,}000\ (-2/4{,}000) - 5D = 5$$

Thus, D = 3/10 or 30 percent, which agrees with the value obtained from the final tableau.

This means that a unit increase in F (funds) would result in a 30 percent return in terms of the savings to the system.

REFERENCE

7. Pinnell, C., and Satterly, G., "Systems Analysis Technique for the Evaluation of Arterial Street Operation." Presented at ASCE Convention, Detroit, Mich. (Oct. 1962).