Dynamic Phenomena in Layered Structures

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Much experimental and theoretical work is being done in France related to the application of vibration techniques to road structures. This paper is concerned with the approach through the theory of elasticity. An analysis is offered of some special cases of dynamic phenomena in layered systems, so as to bring forth some simple and general rules, and allow a rough interpretation of the experimental results. Among these phenomena are (a) in a semi-infinite medium, the dynamic effects due to a transient vehicle (role of the Rayleigh wave), (b) free waves in layered systems (namely, one layer on subgrade) when the Lamé's parameters of the two materials are in some characteristical ratios (Rayleigh waves), and (c) one or two layers on subgrade (some remarks on Love waves). The results thus obtained can be explained by the reflection and refraction theory of elastic waves allowing generalization to any member of layers. The paper also discusses some ideas on the relative importance of static and specifically dynamic effects depending on the distance of the source of disturbance.

•THE PROBLEM that arises in the rational design of road structures consists, under given conditions of traffic, subgrade, economics, etc., is the choice of the most suitable among the available materials, and the evaluation of the layer thicknesses. To solve this problem, it is necessary:

1. To know which stresses take place in a theoretically given structure. Since the work of Jeuffroy and Bachelez (1) different methods (2, 3) have been used in France, that of Lattes et al., (4) being more adapted to the unavoidable numerical computations.
2. Conversely, to carry out tests on existing roads so as to find out which values of elastic parameters can be associated with each material used. The testing implies knowledge of some general rules relating the response of a road to the applied load.

The present paper deals with both these points, using the theory of free waves in layered structures to show how they permit specific description of dynamic effects, and then describing some of their general features.

SEMI-INFINITE ELASTIC MEDIUM

Transient Load—Theoretical Approach

The study of a load traveling at uniform speed was undertaken at first with the idea of quantitatively comparing the resulting displacements and those of Boussinesq's solution in the corresponding static problem. Because it has been possible to give a completely explicit solution, and to find a material suitable for the experiment, this special case is dwelt on, although it is not characteristic of layered structures.

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The method used to give the theoretical result is the well-known "superposition of solutions." Only the principal features of the analysis are indicated here, inasmuch as further details are given elsewhere (5, 6).

The problem is governed by the following equations:

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{D} + \mu \Delta \vec{D} = d \frac{\partial^2 \vec{D}}{\partial t^2} \tag{1}$$

(equation of elasticity) in which λ and μ are Lamé's parameters, d is the density of the medium, $\vec{D}$ is the displacement vector, and t is time.

$$\sigma_{3j} = \lambda \operatorname{div} \vec{D} \delta_{3j} + \mu \left(\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} \right) \tag{2a}$$

$$= -X_j \, \delta (x_1 - Vt, x_2) \text{ if } x_3 = 0 \tag{2b}$$

where Eq. 2a is the elastic relationship between stress and strain and Eq. 2b represents the boundary conditions, with x_1, x_2, x_3 being space coordinates (with origin 0 on the horizontal boundary plane, $0x_3$ vertical pointing downwards, $0x_1$ in the direction of the speed $\vec{V}$), u_1, u_2, u_3 the displacement components, X_1, X_2, X_3 the components of the assumed concentrated applied force, $\delta (x_1 - Vt, x_2)$ is Dirac's distribution, and δ_{3j} is Kronecker's delta.

In the following analysis x'_1 represents $x_1 - Vt$.

Note that

$$\delta (x'_1, x_2) = \frac{1}{4\pi^2} \lim_{L \to \infty} \iint_{V(L)} e^m \, da_1 \, da_2$$

in which $V(L) = a_1^2 + a_2^2 < L^2$

and $M = i (a_1 x_1^1 + a_2 x_2)$

so that, if the displacements corresponding to

$$\sigma_{3j}^{(e)} = -\frac{x_j}{4\pi^2} e^m \quad \text{for } x_3 = 0 \tag{3}$$

are known, the solution of the problem is obtained by the superposition

$$\lim_{L \to \infty} \frac{1}{4\pi^2} \iint_{a_1^2 + a_2^2 < L^2} \text{- - -} \, da_1 \, da_2$$

inasmuch as all the equations of the problem are linear.

If the elementary displacement is written in the form

$$\vec{D}^{(e)} = \overrightarrow{D_g} + \overrightarrow{D_r} \qquad \text{with} \quad \begin{cases} D_g = \text{grad } g \\ D_r = \text{rot* } \vec{R} \end{cases}$$

it is possible to choose:

$$g = -i\, C\, e^{i(a_1 x'_1 + a_2 x_2 + i\, c_g\, x_3)}$$

which gives $\overrightarrow{D_g}$, with

$$c_g = \left(a_1^2 + a_2^2 - d\, \frac{a_1 V^2}{\lambda + 2\mu}\right)^{1/2} > 0$$

assuming $V^2 < \frac{\lambda + 2\mu}{d}$.

Also,

$$\overrightarrow{D_r} = \begin{cases} c_r\, A_1\, e^{i(a_1 x'_1 + a_2 x_2 + i c_r x_3)} \\ c_r\, A_2\, e^{i(a_1 x'_1 + a_2 x_2 + i c_r x_3)} \\ i(a_1 A_1 + a_2 A_2)\, e^{i(a_1 x'_1 + a_2 x_2 + i c_r x_3)} \end{cases}$$

with

$$c_r = \left(a_1^2 + a_2^2 - d\, \frac{a_1^2 V^2}{\mu}\right)^{1/2} > 0$$

assuming $V^2 < \frac{\mu}{d}$. It should be noted that div $\overrightarrow{D_r} = 0$, which gives the term $+ i(a_1 A_1 + a_2 A_2)$.

In calculating the corresponding elementary stresses $\sigma_{3j}^{(e)}$ by Eq. 2a, and comparing the results with Eq. 3 for $x_3 = 0$, a linear system appears which determines C, A_1, A_2 (note that the use of c_g and c_r allows the elimination of the explicit use of λ, although this coefficient still plays a part through c_g).

This system is

$$2\, c_g\, a_1 C + (a_1^2 + a_2^2)\, A_1 + a_1 a_2 A_2 = \frac{X_1}{4\pi^2 \mu} \tag{4a}$$

$$2\, c_g\, a_2 C + a_1 a_2 A_1 + (a_2^2 + c_r^2)\, A_2 = \frac{X_2}{4\pi^2 \mu} \tag{4b}$$

*Curl.

$$(a_1^2 + a_2^2 + c_2^2)\ C + 2c_r\ (a_1 A_1 + a_2 A_2) = -\frac{i x_3}{4\pi^2 \mu} \tag{4c}$$

The determinant of this system, to within a positive factor, is

$$f(\zeta^2) = 4\left(1 - \frac{\zeta^2}{\Omega_g^2}\right)^{1/2}\left(1 - \frac{\zeta^2}{\Omega_r^2}\right)^{1/2} - \left(2 - \frac{\zeta^2}{\Omega_r^2}\right)^2, \quad \left\{\begin{array}{l} \zeta^2 = \dfrac{a_1^2 V^2}{a_1^2 + a_2^2} \\ \Omega_g^2 = \dfrac{\lambda + 2\mu}{d} \\ \Omega_r^2 = \dfrac{\mu}{d} \end{array}\right.$$

with $f(V_R^2) = 0$, V_R being the velocity of the Rayleigh wave,
Because $f(\zeta^2)$ is a factor in the denominator of C, A_1, A_2, and of the integrand in

$$\vec{D} = \lim_{L \to \infty} \iint_{a_1^2 + a_2^2 < L^2} D^{(e)}\, da_1\, da_2$$

$$\vec{D} \longrightarrow \infty \text{ if } V \longrightarrow V_R.$$

However, $\vec{D}$ can be calculated further, if $V < V_R$ (which is realized in all practical cases), by using the following change of variables:

$$x_1 = r\cos\alpha \qquad a_1 = \rho\cos\theta$$
$$x_2 = r\sin\alpha \qquad a_2 = \rho\sin\theta$$

The result, u_3, corresponding to a normal load (i.e., $X_1 = X_2 = 0$) (see Mandel and Avramesco (6) for generalization to u_1, u_2; X_1, $X_2 \neq 0$; non-concentrated load) is

$$u_3 = \frac{X_3}{2\pi\mu r}\ \frac{\xi^2}{\Omega_r^2}\ \frac{\left(1 - \dfrac{\xi^2}{\Omega_g^2}\right)^{1/2}}{f(\xi^2)} \quad \text{with } \xi = V\sin\alpha$$

of which an approximate value, when $\xi/\Omega_r \ll 1$, may be obtained from

$$u_3 = \frac{x_3}{4\pi\mu r}\ \frac{\lambda + 2\mu}{\lambda + \mu}\left[1 + \frac{\xi^2}{4\Omega_r^2}\left(1 - 3x + \frac{2}{1x}\right) + \ldots\right]$$

These results are discussed under "Dynamic Effects" and compared with those of the experiment described in the following.

Transient Load—Experiment

Realization of an experiment giving evidence of the interference of Rayleigh's wave velocity faces two main difficulties: (a) On ordinary materials, the speeds that are to be reached are much too great; and (b) It is necessary to follow the phenomenon under a moving load.

Materials on which V_R is very low are found easily enough: they are rubbers and gelatines. Rubbers are discarded as really viscoelastic or nonhomogeneous.

An ordinary gelatine can be chosen, however, that is simple to prepare and for which different Young's moduli can be obtained by adding more or less water to a kind of glue.

The second difficulty is solved by the use of a special type of container (Fig. 1) which rotates while a ball attached to the end of a fixed rod runs on the surface of the gelatine. The gelatine is some 15 cm thick, which is sufficient to considering the mass as "infinitely deep." A small electric motor drives the container at the various desired speeds.

This experiment, as well as that described under "Free Waves," was suggested by Mr. Habib, Associate Director, Solid Mechanics Laboratory, Ecole Polytechnique, Paris.

As the speed of rotation increases, the part of the surface deformed under the ball becomes more and more important, although rather slowly, spreading especially in the direction perpendicular to $\vec{V}$. When the speed V_R is reached a shock wave appears, slightly distorted because the speed is not the same at the different distances from the center of rotation. If V is further increased, the well-known aspect of Mach's angle appears (Fig. 2), with

$$\varphi = \arcsin \frac{V_R}{V}$$

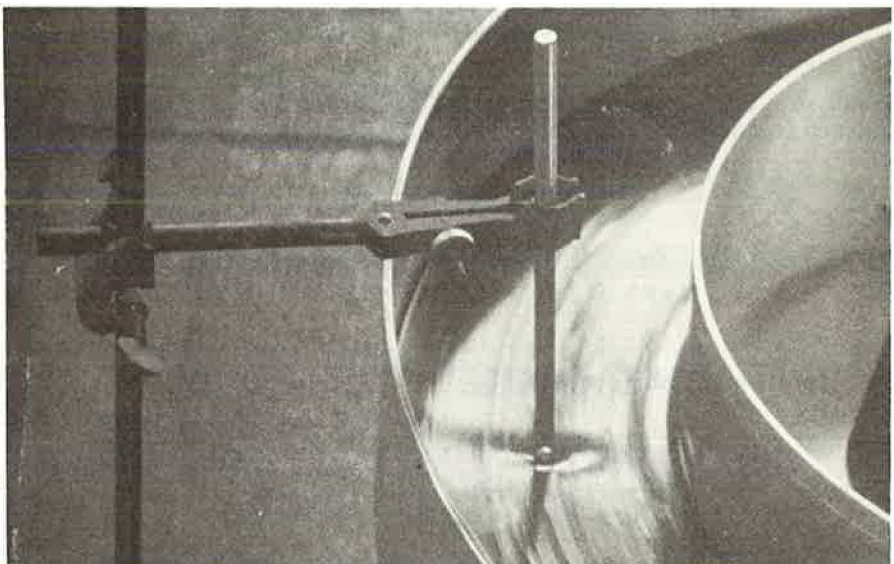

Figure 1.

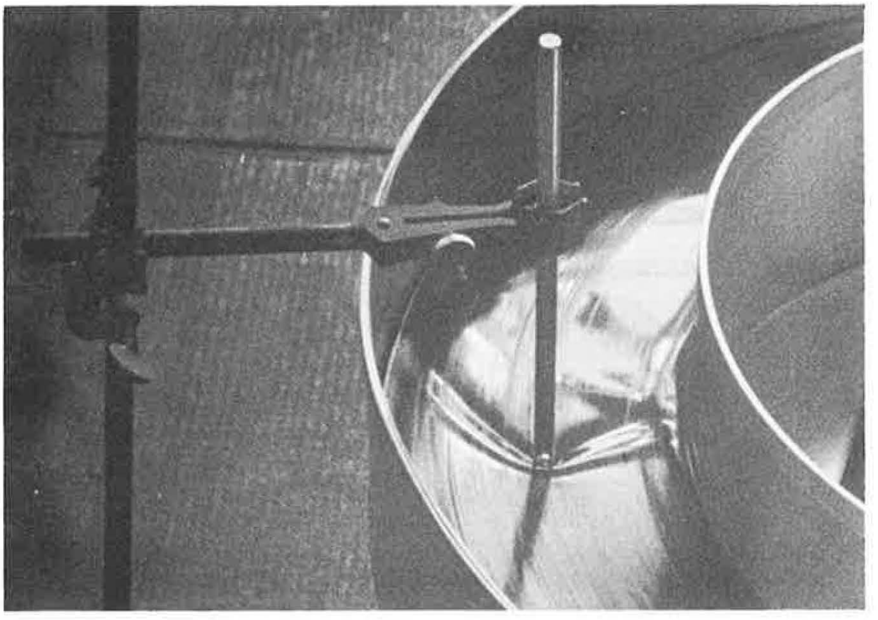

Figure 2.

The velocity, V_R, is of the order of 1.5 m/sec in the gelatine used. The measures of its Young's modulus by static methods give fairly good corresponding values to the V_R observed.

Dynamic Effects

First, the agreement between theory and experiment, except for the a priori evident interference of viscosity, should be noted. That is, if $V \longrightarrow V_R$, $\vec{D}$ is greatly increased, but does not become infinite. The experiment also shows that the important variations of $\vec{D}$ take place when V is not much less than V_R.

Figure 3 gives the theoretical results concerning u_3 for the Poisson's ratio $\nu = \frac{1}{4}$. For instance, an increment of some 7 percent of the static displacement is found for $V = 0.75\ \Omega_r$ (V_R is of the order of $0.9\ \Omega_r$). The approximate value given at the end of the section on "Transient Load—Theoretical Approach," which is fairly accurate in practical analysis, shows the small influence of the Poisson's ratio: the factor of $\xi^2/4\Omega_r^2$ is 3 for $\nu = \frac{1}{2}$ $(x = 0)$ and $\nu = \frac{1}{4}$ $(x = \frac{1}{3})$, and varies little for $\nu \in [0, \frac{1}{2}]$.

It must be remembered that on ordinary subgrades the Rayleigh wave velocity is very seldom less than 100 m/sec (360 km/hr).

It is then possible to consider either that (a) no effect due to such a movement of a load will appear, and that this is some evidence that specifically dynamic effects may not be important in stress distributions, or that (b) this is evidence of the role of free waves, producing important deformations far from the source of disturbance.

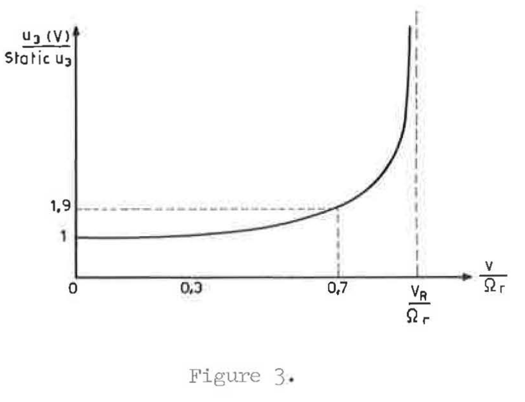

Figure 3.

It is certainly of mainly academic interest by now to analyze this generation of a Rayleigh shock wave. But in layered systems, Rayleigh wave velocities are not always so large, and it is important to observe how, in this case, dynamic effects are closely connected to the existence of free waves.

The problem of the transient load on a semi-infinite homogeneous elastic medium has been solved. As for the vibrating load, different attempts have been made (2, 7, 8) that are not satisfactory, at least because they imply difficult computations or an implicit form of the results. However, many experiments have been carried out. They use two different kinds of apparatus—light or heavy—permitting one or several results (velocity, strains). These experiments are the same for a semi-infinite medium and for several layers, and the apparatus and methods are described in under the heading "Attempts at Generalization."

A complete program of experiments is being carried out with a light vibrator at the Polygone d'Essais du Laboratoire des Ponts-et-Chaussees in Rouen. The purpose of the first tests has been to determine some fundamental characteristics of the "subgrade"; that is, if it can be considered to be (a) homogeneous in both the horizontal and vertical directions, and (b) "semi-infinite" (i.e., if important disturbances are created by the underlying natural soil or by the cement blocks of the pit banks). These tests are important because they give evidence as to whether or not the theory of the Rayleigh wave can be applied.

A silty clay 1.5 m deep gives results as near to the theoretical result (V_R = constant for the different frequencies) as can be wished, for a chosen point at the surface. The horizontal homogeneity also can be considered as very good. No disturbances are created by the pit banks, inasmuch as the results are identical along all the axes of measurements (Fig. 4). The points at which the phase is the same as, or the opposite of, a certain reference, give the sequence of half wavelengths.

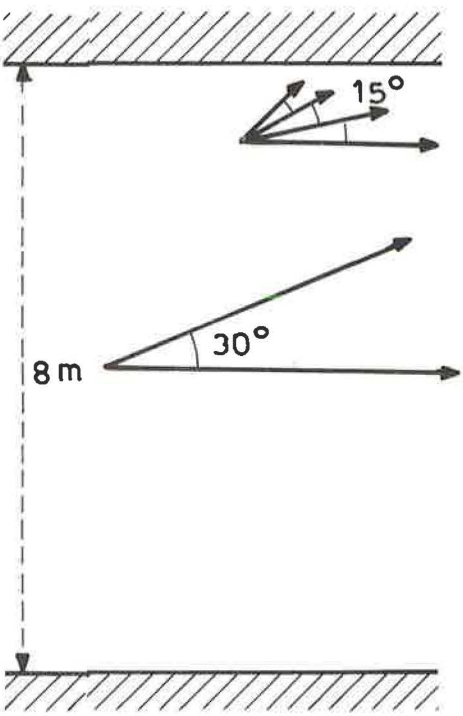

Figure 4.

FREE WAVES

One Layer on Subgrade—Rayleigh Waves

Many theoretical analyses and experimental computations have been carried out on the subject of free elastic waves in layered systems (9, 10). For the special case of one layer on a semi-infinite homogeneous subgrade, the author previously has provided an analysis (11). Some results of this last work are first reviewed, then are somewhat generalized in a following section.

The symbols of the previous section are still used, with λ and μ being Lamé's parameters, d the density of the medium, and

$$\Omega_g = \left(\frac{\lambda + 2\mu}{d}\right)^{1/2} \qquad \Omega_r = \left(\frac{\mu}{d}\right)^{1/2}$$

In the following the prime (′) refers to the subgrade. The equation of elasticity (Eq. 1) still holds in the different media and there is "perfect friction" on the horizontal plane between the surface, L, and the subgrade, L′; that is,

$$\left.\begin{array}{l} \sigma_{3j}\,(L) = \sigma_{3j}\,(L') \\ \mu_i\,(L) = \mu_i\,(L') \end{array}\right| \begin{array}{l} \text{on this plane} \\ (i,\ j\ =\ 1,2,3) \end{array} \qquad (6)$$

V is now the velocity of a free wave, measured at the free surface S of L, and N and l are the associated frequency and wavelength, respectively. It must be borne in mind that a free wave is (a) from the mathematical point of view, a solution of the equations of elasticity (Eq. 1) and continuity (Eq. 6) corresponding to $\sigma_{3j} = 0$ on S, and (b) from the physical point of view, a wave transmitted by the media in such a way that there is no downward-running energy. It also must be remembered that in layered media the wavelength of a free wave depends on its velocity—the frequency being then given by $N = V/l$—which is not true for a semi-infinite homogeneous medium (see "Semi-Infinite Elastic Medium,") in which V = constant = V_R for any l.

If the reduced shear moduli are the same for L and L$'$ (that is, $\mu/d = \mu'/d'$) it is possible to show that $V \epsilon [V_R, V'_R]$, in which V_R and V'_R are, respectively, the Rayleigh wave velocity in a semi-infinite homogeneous medium on its own with the same parameters as L and L$'$. Furthermore,

$$f\,(V_R^2) = 0 \quad f'\,(V'^2_R) = 0$$

(see II. A for definition of f).

b) - as a priori evident

$$l \longrightarrow 0 \quad \Longrightarrow \quad V \longrightarrow V_R$$

$$l \longrightarrow \infty \quad \Longrightarrow \quad V \longrightarrow V'_R$$

which simply means that if the wavelength is very small (resp. great), it appears as if L (resp. L$'$) interfered alone and were semi-infinite.

It is remarkable that in this special case V lies between the two ordinary Rayleigh wave velocities, a simple result which unfortunately does not hold in general, as shown in the following.

If it is assumed that $\lambda' = \mu' = 0$, Lamb's solution is obtained (12) with, again,

$$l \longrightarrow 0 \quad \Longrightarrow \quad V \longrightarrow V_R$$

But there is at least one free wave for any V, one if $V^2 < \mu/d$, an infinity if $V^2 > \mu/d$.

When V and l tend to infinity the asymptotic frequencies are

$$N_g = k\,\frac{\Omega_g}{2\,p}$$

$$N_r = k\,\frac{\Omega_r}{2\,p}$$

in which k is any positive integer and p is the thickness of L.

The interpretation of this last result is simple: if V is the measured velocity of a free wave, there is between the ray perpendicular to the corresponding S = wave (resp. P = wave) transmitted in the medium, and S an angle

$$\varphi = \text{arc cos } \frac{\Omega_r}{V}$$

$$\left(\text{resp.} \quad \alpha = \text{arc cos } \frac{\Omega_g}{V}\right)$$

$$V \longrightarrow \infty \Longrightarrow \varphi \longrightarrow \frac{\pi}{2} \left(\text{resp. } \alpha \longrightarrow \frac{\pi}{2}\right)$$

and the analysis is then that of a stationary shear wave (resp. P-wave) perpendicular to the limiting free planes:

$$k\,\frac{1}{2} = p \quad \text{and since} \quad 1 = \frac{\Omega_R}{N_R} \left(\text{resp. } \frac{\Omega_g}{N_g}\right)$$

$$N_r = k\,\frac{\Omega_r}{2p} \left(\text{resp. } k\,\frac{\Omega_g}{2p}\right)$$

If λ' and $\mu' \longrightarrow \infty$,

$$1 \longrightarrow 0 \Longrightarrow V \longrightarrow V_R$$

but here

$$V \geq V_R$$

with one solution 1 if $V^2 < \mu/d$, and an infinity if $V^2 > \mu/d$.

When V and 1 $\longrightarrow \infty$, the asymptotic frequencies are

$$N_g = (2k + 1)\,\frac{\Omega_g}{4p}$$

$$N_r = (2k + 1)\,\frac{\Omega_r}{4p}$$

which stand between those of the case examined in the previous section, as is evident because the lower limiting plane is now fixed.

The experiments of the transient load have been carried out for this case under conditions similar to those described under "Transient Load—Experiment." Some important points are as follows:

1. The theoretical approach is analogus to the method discussed in an earlier section, but it only leads to integrals, representing the displacements, which it so far has not been possible to calculate more explicitly.
2. A first experiment in which the thickness of gelatine varies under the running ball allows comparison of the corresponding phenomena and shows that several attenuated waves appear behind the first shock wave when $V \gg V_R$ and when the thickness is small enough.
3. The second experiment, with a small thickness of gelatine, shows that these typical attenuated waves are reflections of the initial one on the rigid surface of the "subgrade" (the container), under the gelatine.

Attempts at a Generalization

The interpretation of these rather different results is given by applying the general theory of reflection and refraction of elastic waves, the elements of which have been set forth by Sommerfeld (13) and with more or less further details elsewhere (9, 10. 11). This theory allows a generalization to more complex problems.

The results are given below without proofs, although these are sometimes quite simple:

1. Let J be the incident wave on a plane S; P a reflected P-wave, P′ a refracted P-wave; S a reflected S-wave, S′ a refracted S-wave; Ω_g, and Ω_r the P- and S-wave velocities in L (resp. Ω'_g and Ω'_r) in which propagate J, P, and S (resp. P′ and S′). The angles perpendicular to S and perpendicular to the wave surface) are, respectively, α, α', φ, φ' for P, P′, S, S′.

2. If V is the velocity of a wave, measured on S,

$$\frac{\sin\alpha}{\Omega_g} = \frac{\sin\alpha'}{\Omega'_g} = \frac{\sin\varphi}{\Omega_r} = \frac{\sin\varphi'}{\Omega'_r} = \frac{1}{V}$$

3. To J, either P- or S-wave, there are four corresponding waves P, P′, S, S′.
4. If, for instance, $\Omega_g < \Omega'_r$, to a penetrating P-wave in L there may correspond in L′ either a penetrating S-wave if $\sin\alpha < \Omega_g/\Omega'_r$ (α being the angle of incidence) or an attenuating S-wave if $\sin\alpha > \Omega_g/\Omega'_r$.
5. When to a penetrating incident wave in L there are corresponding attenuating waves in L′, no energy is transmitted from L to L′.
6. Both fixed or a free surface are perfectly reflecting.

It is now possible to explain why in two previous sections free waves are found for any $V > \Omega_r$: in such cases no energy is transmitted from L to L′. On the contrary, another section shows that to any V there can be a corresponding penetrating S′, which implies loss of energy from L, and is therefore contradictory with the definition of a free wave.

This result can be generalized as follows:

If Ω'_r is the velocity of S-waves in the lower medium of an elastic stratified structure, Σ, no free waves can exist but those for which $V < \Omega'_r$.

A first corollary is that if Ω'_r is the lowest S-wave velocity of Σ all free waves are attenuating ones.

A second corollary is that, since Love waves are penetrating in at least one of the layers, no Love waves can exist if Ω'_r is the lowest S-wave velocity of Σ.

The range of velocities where the free waves of a road structure will be encountered, can therefore be restricted and an attempt could be made to develop it further—for instance, in order to explain why it is $[V_R, V'_R]$ only, when the reduced shear moduli are the same for L and L′, etc.

In fact, the usefulness of such a limitation is not evident, for even the basic theorem is not practically true; the experiments show, on the contrary, that velocities $V \gg \Omega'_r$ can be observed. Whether these correspond to free waves or not is discussed in the following.

Experiments on vibration techniques applied to testing of road structures have been carried out in two distinct ways:

1. The method worked out by the Shell Laboratory of Amsterdam (14) uses large loads (a few tons) and low frequencies (up to some 50 hertz). The results of the different measurements made in an experiment have not really been given a clear and general interpretation prior to this. The opinion of the author, who followed the results obtained with the Shell machine hired by the French Laboratoire Central des Ponts-et-Chaussees, is that the created waves are a rather complex interference of free and forced

waves; the problem of the vibrating load has not yet been solved even for a semi-infinite homogeneous medium, much less for a stratified structure. It is not to be expected, therefore, that experiments should receive a simple interpretation, and a better knowledge of the theoretical phenomena would make possible much more use of them.

2. The light apparatus used by the British Road Research Laboratory (15) naturally involves some difficulties. For instance, it is considered by Jones (15) that when the lower medium, L', has a low Young's modulus and a Poisson's ratio near 0.5, an equation giving correct results may be obtained by replacing L' by a liquid in the theoretical structure, without it being denied that this meets no clear justification. However, the information thus obtained (a) is in more than qualitative agreement with other measures and predictions; (b) has proved itself of great value as a testing method on road structures; and (c) is to the knowledge of the author the only one displaying such good qualities for dynamic testing and experiments on roads.

CONCLUDING REMARKS

Other Experiments

Free waves provide the best methods for understanding the general outline of dynamic behavior in road structures, as has been emphasized throughout this paper. In fact, other experiments using free waves can be imagined and have been tried in France. The loadings are in the form of impacts, produced for instance by means of a heavy load falling on a kind of stiff spring. It is difficult to foretell the value of these experiments, as they have no connection whatsoever with the ordinary knowledge of layered structures, and so call for an entirely new and distinct interpretation.

Other Problems

In the case of Lamb's solution (12) the free wave velocity can be very low; in reality λ' and μ' are never null; but, if the lower medium is "soft enough" (low Young's modulus and Poisson's ratio really different from 0.5), the vehicle speed V_s may be great enough to bring about very important dynamic effects. This is especially true if one certain frequency is associated to V_s (ordinary loads being in fact both transient and vibrating). It may be that such an occurrence only seldom arises; however, a road does not fail only by ordinary light traffic, but also by such dangerous (and, if the road is well built, rare) interactions.

It must also be borne in mind that:

1. Free waves extend the field of interaction of the load; they can exist, by definition, anywhere under a free surface.
2. They certainly exist alone at the free surface far enough from the load, inasmuch as the remainder of the dynamic energy must have disappeared in the subgrade.
3. They correspond to the "resonances" of finite media.

These are some reasons why dynamic phenomena and free waves are so closely connected: because free waves, as easily propagating, are both the most dangerous and the most easily observed. Exactly what part they play in the real stresses and failures of roads cannot yet be described quantitatively.

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