Equations of Failure Stresses in Materials With Anisotropic Strength Parameters

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The Mohr-Coulomb equations of the failure stress and Prandtl's equation of the bearing capacity are extended to include the case of a medium with anisotropic cohesion and anisotropic angle of internal friction.

•THE RUPTURE theory conventionally applied to soils and asphaltic concretes is that of Mohr-Coulomb, according to which rupture occurs when the major (σ_1) and minor (σ_3) principal stresses determine a stress circle tangent to the strength line of the material. The latter is, at first approximation, a straight line and yields the following two strength parameters: cohesion C and angle of internal friction ϕ.

Normally these parameters are regarded as isotropic, although as previously shown (1 - 3) this is not always the case in asphaltic concretes. This is especially evident in materials where density is acquired by means of mechanical compaction, whether in the laboratory or in the field. In such compaction the compactive load is transmitted to the material in one direction only, a fact which suffices for the assumption of anisotropy of the resulting structure and hence of strength parameters as well. In these circumstances the failure force which acts parallel to the direction of compaction is greater than that perpendicular to it.

With regard to dense asphaltic concretes it was suggested (1, 2) that the anisotropy of their strength is due solely to that of the cohesion, while the friction angle remains isotropic. This assumption is reasonably accurate when the asphaltic concrete is, for example, composed of cubic aggregates. However, with aggregates of elongated shape it has been shown (3) that both the cohesion and the internal friction angle are anisotropic. This fact necessitates extension of the conventional strength theory to include the case of anisotropy of both strength parameters.

THE FUNCTION DESCRIBING THE VARIATION C AND ϕ

Theoretical calculation of the failure stresses in a material with anisotropic friction angle and cohesion necessitates knowledge of the values of the cohesion and angle of internal friction in every direction.

Where the angle of friction is isotropic and the cohesion anisotropic, a function has been proposed for this variation (2). This function, although a theoretical one, satisfies tests results which show that the cohesion has two major values: C_{max}, which comes into play when the principal stress is parallel to the direction of compaction; and C_{min}, which comes into play when it is perpendicular to the direction of compaction.

The function is:

$$C = C_{max} - \left(C_{max} - C_{min}\right) \sin^2 \bar{\psi} \tag{1a}$$

with

$$\bar{\psi} = \beta_i - \left(45° - \frac{\phi}{2}\right) = \beta_j + \left(45° - \frac{\phi}{2}\right) \tag{1b}$$
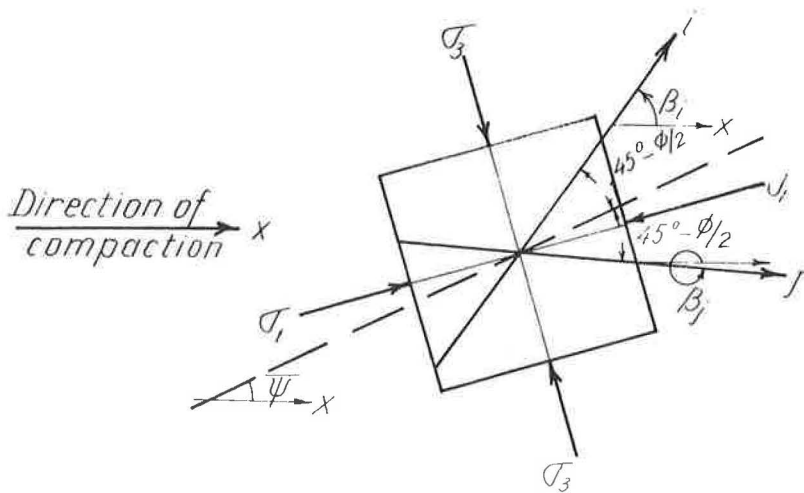
Figure 1. Slip planes in an element with anisotropic strength parameters.

β_i and β_j being the inclination of the failure planes i and j, respectively, as measured from the x-direction (that of compaction; see Fig. 1).

The above proposed function satisfies the limiting conditions. When $\bar{\psi} = 0$, i.e., the direction of the major principal stress is parallel to the direction of compaction, $C = C_{max}$. When $\bar{\psi} = 90°$, i.e., the direction of the major principal stress is perpendicular to the direction of compaction, $C = C_{min}$. For an intermediate value of $\bar{\psi}$, the cohesion has an intermediate value between C_{max} and C_{min}. The function proposed allows for only an initially small reduction in C from C_{max}, as the value of $\bar{\psi}$ increases from 0, and hence is considered to approximate the true behavior closer than another assumption such as a linear variation.

Experimental data are not yet available that define more closely the function of actual variation of C between the two extreme values.

If the value of ϕ varies with the orientation of the failure plane, a function similar to the foregoing can also be proposed:

$$\tan \phi = \tan \phi_{max} - (\tan \phi_{max} - \tan \phi_{min}) \sin^2 \bar{\psi} \tag{2}$$

The relation between the maximum and minimum values of cohesion and friction may be defined as:

$$\frac{\tan \phi_{max}}{\tan \phi_{min}} = m_\phi \tag{3a}$$

$$\frac{C_{max}}{C_{min}} = m_c \tag{3b}$$

For the purpose of further development in this paper, the special case is analyzed of $m_c = m_\phi = m$. This case is represented geometrically by all strength lines originating as straight lines from a single point O' (Fig. 2), because in this case $C \cot \phi$ is a constant independent of $\bar{\psi}$.
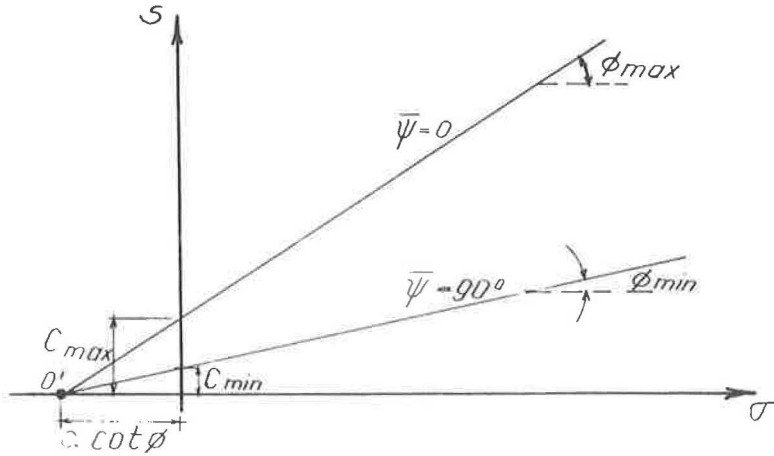
Figure 2. Strength lines diagram.

THE FAILURE STRESS EQUATIONS

The general formulation of the shear strength law for a medium with anisotropic cohesion is

$$S = C + \sigma_n \tan \phi \tag{4}$$

in which

S = the shear strength;
C = cohesion;
ϕ = angle of internal friction (dependent on the orientation of the failure plane); and
σ_n = stress perpendicular to the failure plane.

From the above law it is possible to obtain the values of σ_x, σ_y, τ_{xy} determining the state of stress of a point in plane problems. It is convenient, for this purpose, to employ Mohr's stress circle at failure of the point considered. In a Mohr circle of given radius R (Fig. 3) failure occurs in the plane in which the conditions

$$\frac{dS}{d\beta} = \frac{d\tau}{d\beta} \tag{5a}$$

$$S = \tau \tag{5b}$$

are satisfied.

However, since $\tau = R \sin 2(\beta - \psi)$, Eqs. 5a and b reduce in effect to the following:

$$\frac{dS}{d\beta} = \frac{2S}{\tan 2(\beta - \psi)} \tag{6}$$

ψ being the angle between the x-axis and the plane of the minor principal stress.

Eq. 6 prescribes the condition for the inclination of the failure plane β_i for a Mohr circle of given radius R. Geometrical representation of this equation is obtained with segment a of Figure 3 given by
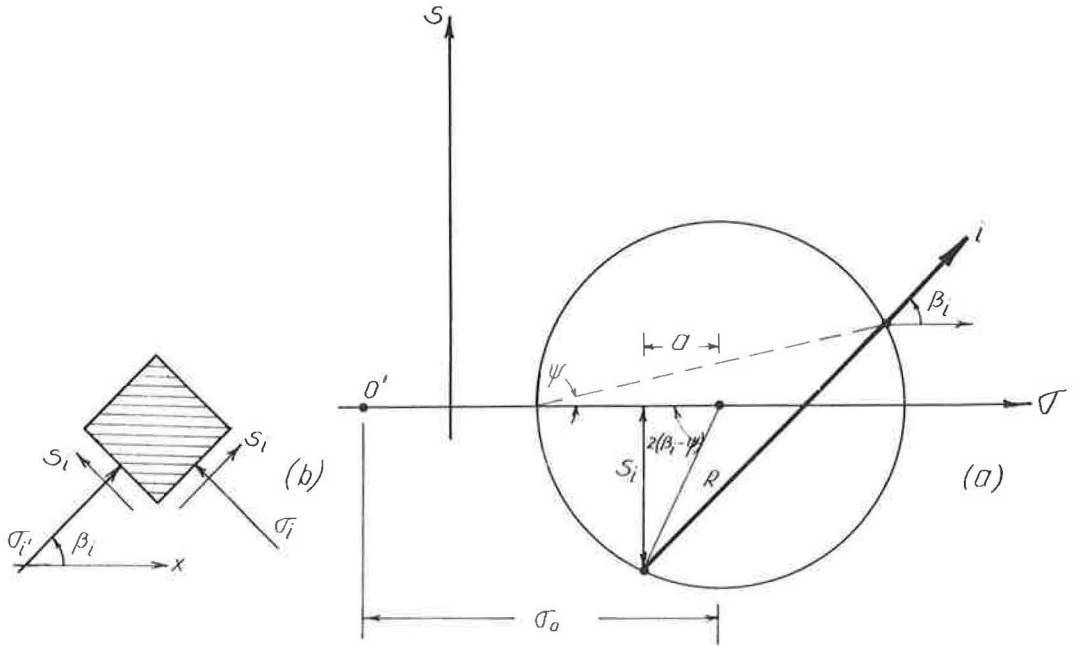
$$a = \frac{1}{2} \frac{dS}{d\beta} \tag{7}$$

Figure 3. (a) Stress diagram; (b) stresses parallel and perpendicular to slip line direction.

In order to determine a, S must be differentiated with respect to β.
The relation between S and β is

$$S = \left[\sigma_0 - R \cos 2(\beta_i - \psi) \right] \tan \phi \tag{8}$$

where σ_0 represents the distance between O′ and O (see Fig. 3).
Differentiation of Eq. 8 yields

$$a = R \sin 2(\beta_i - \psi) \tan \phi + \frac{1}{2} \left[\sigma_0 - R \cos (\beta_i - \psi) \right] \frac{\partial \tan \phi}{\partial \beta} \tag{9}$$

It should be borne in mind that in Eq. 8, σ_0 is independent of β since, as already shown, C cot ϕ is constant.
The Mohr circle also gives the relation

$$\sigma_0 - R \cos 2(\beta_i - \psi) = \sigma_0 - a \tag{10}$$

and substitution in Eq. 9 yields

$$a = (\sigma_0 - a) \tan^2 \phi + (\sigma_0 - a) \frac{\partial \tan \phi}{2 \partial \beta} \tag{11}$$

or finally

$$a = \sigma_0 \sin^2 \phi - f \sigma_0 \cos^2 \phi \tag{12a}$$

where

$$f = - \frac{\partial \tan \phi}{2 \partial \bar{\psi}} \cdot \frac{1}{1 + \tan^2 \phi} = - \frac{\partial \phi}{2 \partial \bar{\psi}} \tag{12b}$$

Eq. 12 already includes the substitution

$$\frac{\partial \tan \phi}{\partial \bar{\psi}} = \frac{\partial \tan \phi}{\partial \beta} \left(1 - \frac{\partial \phi}{2 \partial \bar{\psi}} \right) \tag{13}$$

Now, with a known, the stresses can be calculated. The Mohr circle diagram shows that the stresses respectively parallel and perpendicular to the failure plane i (Fig. 3) are:

$$\sigma_i = \sigma_0 - C \cot \phi + a \tag{14a}$$

$$\sigma_i' = \sigma_0 - C \cot \phi - a \tag{14b}$$

$$\tau_i = - (\sigma_0 - a) \tan \phi \tag{14c}$$

Rotating the axes from i/i' to x/y, and substituting a from Eq. 12 the failure stresses σ_x, σ_y, τ_{xy} are obtained as follows:

$$\sigma_x = \sigma_0 (1 + \sin \phi \cos 2 \bar{\psi}) + \sigma_0 f \sin 2 \bar{\psi} \cos \phi - C \cot \phi \tag{15a}$$

$$\sigma_y = \sigma_0 (1 - \sin \phi \cos 2 \bar{\psi}) - \sigma_0 f \sin 2 \bar{\psi} \cos \phi - C \cot \phi \tag{15b}$$

$$\tau_{xy} = \sigma_0 \sin \phi \sin 2 \bar{\psi} - \sigma_0 f \cos 2 \bar{\psi} \cos \phi \tag{15c}$$

It should be borne in mind that in these equations ϕ is a function of $\bar{\psi}$ according to Eq. 2. When ϕ and C are constant, $f = 0$ and Eqs. 15 are identical with their conventional counterparts for the case of isotropic strength parameters. Finally, it should be noted that Eqs. 15 refer to the case of two-dimensional state of stress with the direction of compaction parallel to the x-axis.

MOHR CIRCLE REPRESENTATION

This section deals with the geometry of the Mohr circle at failure of a material with anisotropic strength parameters. In this circle (Fig. 4) PA is the failure plane, i.e., at A the shearing stress equals the shear strength so that the angle AO'O equals ϕ for the given $\bar{\psi}$.

Using Eq. 15, it can be proved that

$$\overline{DP} = \sigma_0 f \cos \phi \sin 2 \bar{\psi} \tag{16a}$$

$$\overline{HD} = \sigma_0 f \cos \phi \cos 2 \bar{\psi} \tag{16b}$$

where in triangle OHP we have

$$\overline{OH} = \overline{OB} = \overline{O'O} \sin \phi \tag{17a}$$
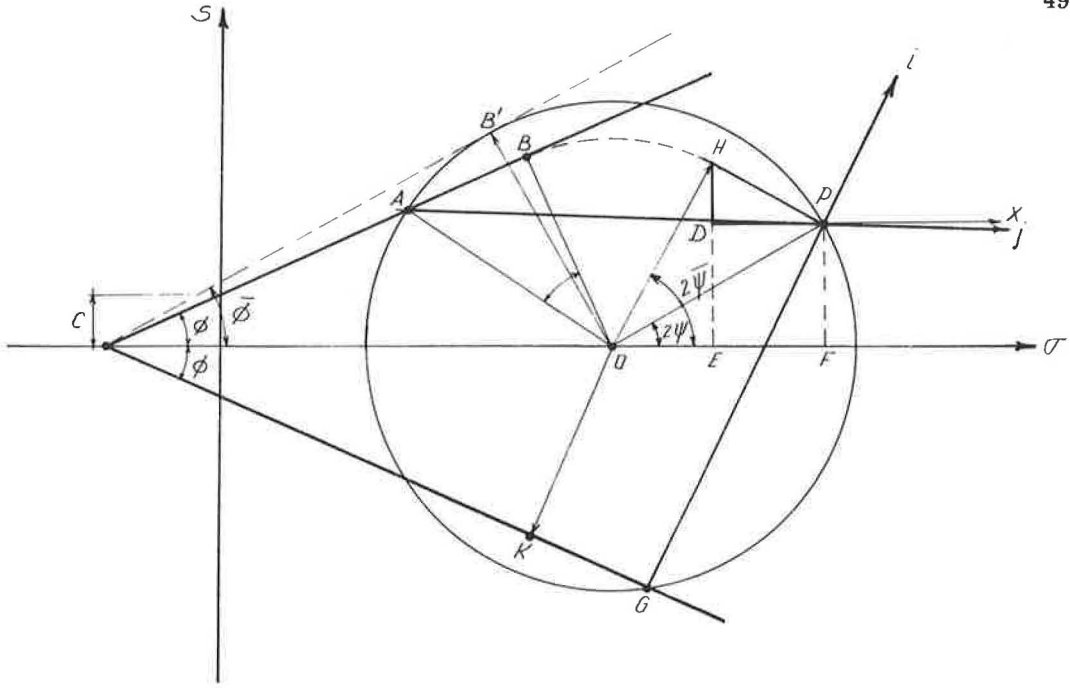
$$\hat{POH} = 2 \bar{\psi} \tag{17b}$$

Figure 4. Mohr circle for anisotropic strength parameters.

$$O\hat{H}P = 90°$$ (17c)

hence

$$\overline{HP} = \sigma_0 \, f \, \cos \phi$$ (18)

Triangles OHG and DAB are congruent, hence

$$O\hat{A}B = 2(\psi - \bar{\psi})$$ (19)

and

$$\overline{AB} = \sigma_0 \, f \, \cos \phi$$ (20)

hence

$$\frac{\overline{AB}}{\overline{O'B}} = f$$ (21)

Obviously, when $f = 0$, A coincides with B, but when $f \neq 0$ the failure circle is tangent not to the strength line O'A but to another line originating in O', its inclination $\bar{\phi}$ being given by

$$\sin \bar{\phi} = \frac{\overline{OB'}}{\overline{O'O}} = \frac{\sin \phi}{\cos 2(\psi - \bar{\psi})}$$ (22a)

or

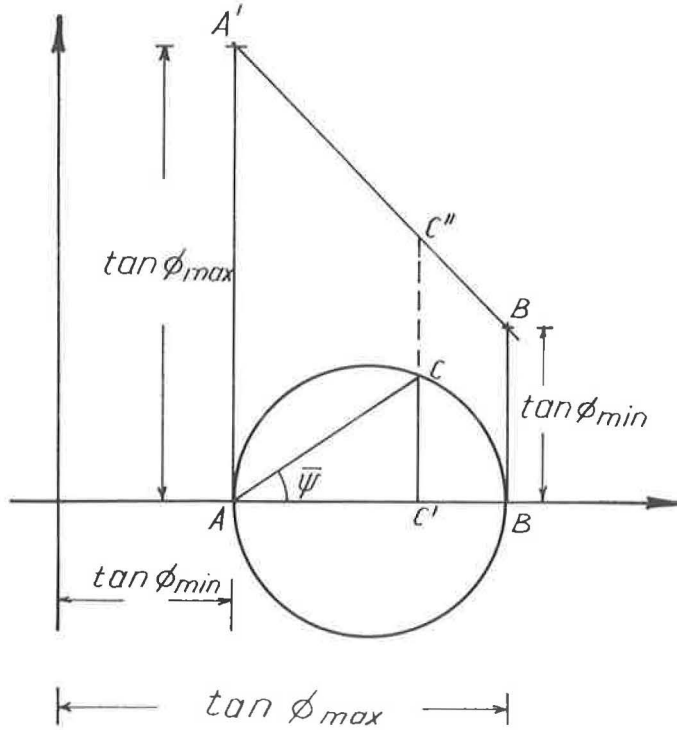Figure 5. Graphical determination of $\tan\phi$ and $\partial\tan\phi/2\partial\bar{\psi}$.

$$\sin\bar{\phi} = \sqrt{\sin^2\phi + f^2\cos^2\phi} \tag{22b}$$

Clearly this failure circle contains a second failure plane, namely the segment $\overline{GP}$, for $APC = 90° - \phi$. Previous derivations show that here again

$$\frac{\overline{KG}}{\overline{O'K}} = f \tag{23}$$

To sum up, a graphical method is given for calculating $\tan\phi$ and $-\dfrac{\partial\tan\phi}{2\partial\bar{\psi}}$ which yields the expression for f. To accomplish this, a circle of diameter $AB = \tan\phi_{max} - \tan\phi_{min}$ (Fig. 5) is drawn. A perpendicular $BB' = \tan\phi_{max}$ is raised at B and a perpendicular $AA' = \tan\phi_{min}$ at A. The straight line $A'B'$ and circle represent the function for $\tan\phi$ and $-\dfrac{\partial\tan\phi}{2\partial\bar{\psi}}$, as follows: from A an inclined line is drawn for the given value of $\bar{\psi}$. CC'' is then the required value of $\tan\phi$ and $C'C''$ that of $-\dfrac{\partial\tan\phi}{2\partial\bar{\psi}}$.

FORMULA OF BEARING CAPACITY

In routine laboratory tests, C and ϕ are determined at $\bar{\psi} = 0$, thus generally yielding C_{max} and ϕ_{max} for a material with anisotropic strength parameters. To calculate the bearing capacity, these values are substituted in Prandtl's formula. This substitution
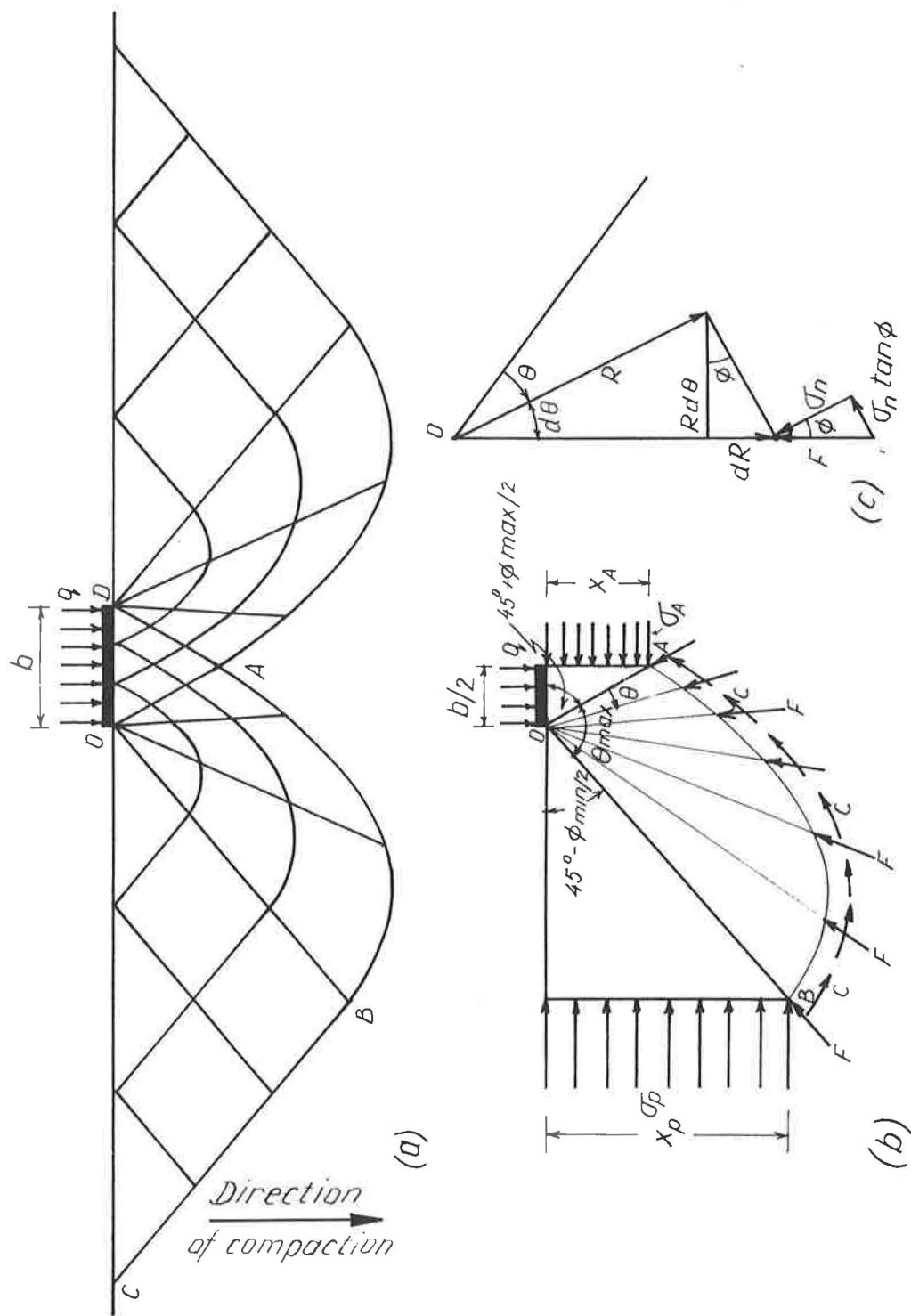
Figure 6. Rupture mechanism for bearing capacity determination.

yields correct results so long as the C and ϕ are isotropic; otherwise, the actual bearing capacity is smaller than the obtained value.

The equations for the anisotropy case are derived below, and demonstrate how important it is to ascertain, prior to calculation, whether the strength parameters are isotropic or anisotropic.

The failure mechanism for this case is shown in Figure 6a, with the direction of compaction parallel to the unit bearing load q. The mechanism consists of (a) triangle OAD, a region of active Rankine shear with angle OAD = $90° - \phi_{max}$; (b) triangle CBO, a region of passive Rankine shear with angle CBO = $90° + \phi_{min}$; (c) sector OAB, a region of radial shear with:

$$\frac{dR}{Rd\theta} = \tan \phi \tag{24a}$$

$$\theta = \phi/2 - \phi_{max}/2 + \bar{\psi} \tag{24b}$$

where R is the radius vector of the failure lines in this latter region. Eq. 24 insures that the resultant of σ_n and $\sigma_n \tan \phi$ (Fig. 6c) in this region passes through O, as in the case of Prandtl's mechanism.

Eq. 24 leads to the following expression for the radius vector in the radial shear region:

$$R = R_0 \exp \left\{ \left[\tan \phi_{max} + \tan \phi_{min} \right] \frac{\bar{\psi}}{2} + \right.$$

$$\left[\tan \phi_{max} - \tan \phi_{min} \right] \frac{\sin 2\bar{\psi}}{4} +$$

$$\left. \left[\tan^2 \phi - \tan^2 \phi_{max} \right] \frac{1}{4} \right\} \tag{25}$$

where R_0 is represented by segment OA in Figure 6a. To find the bearing capacity q, moments are equated about O. The stresses producing moments about O are (see Fig. 6b): σ_A—the active stress, σ_P—the passive stress, the cohesion C along BA (σ_n and $\sigma_n \tan \phi$ along AB do not produce moments about O, since their resultant passes through it), and the bearing capacity q is thus:

$$\sigma_P \frac{x_P^2}{2} + \int_0^{\theta_{max}} C R^2 \, d\theta = \frac{qb^2}{8} + \sigma_A \cdot \frac{x_A^2}{2} \tag{26a}$$

according to Eq. 13, σ_A is given by σ_y for $\bar{\psi} = 0$ and $q = \sigma_1$, i.e.,

$$\sigma_A = q \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}} - \frac{2 C_{max} \cos \phi_{max}}{1 + \sin \phi_{max}} \tag{26b}$$

Similarly, σ_P is given by σ_x for $\bar{\psi} = 90°$ and $\sigma_3 = 0$

$$\sigma_P = \frac{2 C_{min} \cos \phi_{min}}{1 - \sin \phi_{min}} \tag{26c}$$

and substitution and integration yields

$$q = \bar{N}_C \cdot C_{max} \qquad (27a)$$

where

$$1 + \frac{\bar{N}_C}{\cot \phi_{max}} = \left\{ \left[\frac{\sin \phi_{min}}{1 - \sin \phi_{min}} + \frac{1}{1 - \sin \phi_{max}} \right] \right.$$

$$\left. \exp \left[\frac{\pi}{2} \left(\tan \phi_{max} + \tan \phi_{min} \right) - \frac{1}{2} \left(\tan^2 \phi_{max} - \tan^2 \phi_{min} \right) \right] \right\} \qquad (27b)$$

The factor $\bar{N}_C$ can also be given as:

$$\bar{N}_C = F \cdot N_C \qquad (28)$$

where N_C is the bearing factor for a material with isotropic strength parameters, and F a reduction factor accounting for the effect of anisotropic strength. F is plotted in
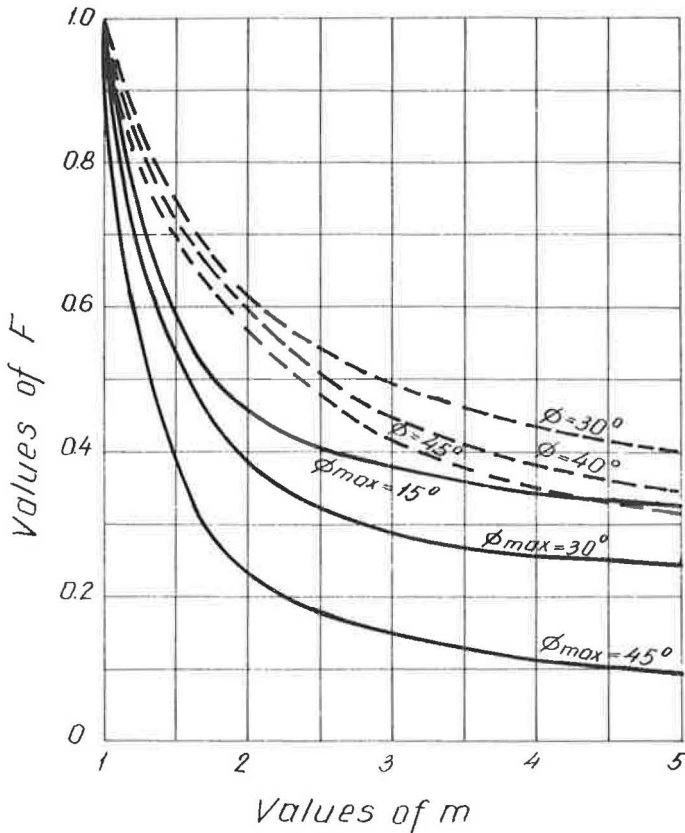

Figure 7. Reduction factor F vs anisotropic strength factor m.

Figure 7 (solid lines) against m for different values of ϕ_{max}; the dashed lines represent the case of anisotropic cohesion and isotropic friction angle (4). Figure 7 shows that the anisotropy effect is decisive, hence the need for ascertaining it. For example, for $\phi_{max} = 45°$ and m = 3, F is 0.15; hence disregard of possible anisotropy will in this case lead to six-fold overestimation of the bearing capacity compared with the result when m is taken into account. Obviously, F will always be smaller in the case of anisotropy of both parameters, compared with its value when the cohesion is anisotropic and the angle of friction isotropic.

CONCLUSIONS

1. The failure stress equation for the case of anisotropic strength parameters involves the term $\partial \phi / \partial \bar{\psi}$; when this term equals zero the equations become identical to the conventional ones.

2. In the case of anisotropic strength parameters the Mohr circle at failure is not tangent to the strength line defined by $S = C + \sigma \tan \phi$.

3. The bearing capacity of materials with anisotropic strength parameters is very much affected by the value of the anisotropic strength factor.

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