Non-Integer Car-Following Models

ADOLF D. MAY, JR., and HARTMUT E. M. KELLER
ITTE, University of California, Berkeley

•SEVERAL authors have originated and developed models to describe the traffic flow on highways. There are at least two approaches to this problem. The microscopic approach, sometimes referred to as the car-following theory, takes as its elements individual vehicular spacing and speed. The macroscopic approach deals with traffic-stream flows, densities, and average speeds. In recent years it has been shown that the two approaches are interrelated.

This paper consists of four major parts. First, a brief background is given of microscopic and macroscopic theories of traffic flow, with special emphasis on their interrelationship. Second, a comprehensive matrix is developed which results in a set of steady-state flow equations, which includes the major macroscopic and microscopic theories. Third, analytical techniques are developed for evaluating the various theories on the basis of experimental data. The last section deals with the investigation of a continuum of non-integer car-following models for the development of deterministic flow models, which describe interrelationships between flow characteristics.

BACKGROUND

The challenge to describe vehicular flow in a microscopic manner led Reuschel (1) and Pipes (2) to formulate the phenomena of the motion of pairs of vehicles following each other by the expression

$$x_n - x_{n+1} = L + S\,(\mathring{x}_{n+1}) \tag{1}$$

This relation can be derived from Figure 1. In this formulation it is assumed that each driver maintains a separation distance proportional to the speed of his vehicle ($\mathring{x}_{n+1}$) plus a distance L. The factor L is the distance headway at standstill ($\mathring{x}_n = \mathring{x}_{n+1} = 0$), including the length of the lead vehicle. The constant S has the dimension of time, and the differentiation of Eq. 1 gives

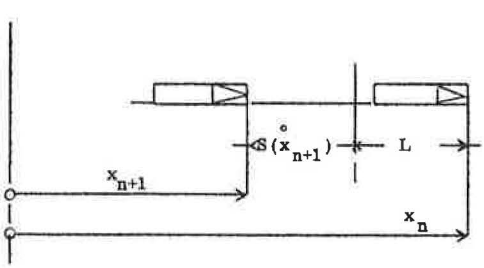

Figure 1. Bases for car-following formula.

$$\overset{\circ\circ}{x}_{n+1} = \frac{1}{S}\,[\mathring{x}_n - \mathring{x}_{n+1}] \tag{2}$$

This differential equation is generally referred to as the basic equation of the car-following models.

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It is the basic stimulus-response relation which was further investigated by a research group from General Motors Corp. (3). Their linear mathematical model showed surprisingly good results for high-density conditions when tested against actual data:

$$\ddot{x}_{n+1}(t + T) = \lambda [\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{3}$$

where T = the time lag of response to the stimulus.

This formulation was refined in 1959 by Gazis, Herman, and Potts (4) by letting the sensitivity factor, λ, be inversely proportional to the distance of separation (distance headway):

$$\lambda = \frac{a_1}{x_n(t) - x_{n+1}(t)} \tag{4}$$

$$\ddot{x}_{n+1}(t + T) = \frac{a_1}{x_n(t) - x_{n+1}(t)} [\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{5}$$

In 1961, Gazis, Herman, and Rothery (5) proposed a more general expression for the sensitivity factor, λ:

$$\lambda = a \frac{\dot{x}^m_{n+1}(t + T)}{[x_n(t) - x_{n+1}(t)]^\ell} \tag{6}$$

The general expression for these microscopic theories thus becomes

$$\ddot{x}_{n+1}(t + T) = a \frac{\dot{x}^m_{n+1}(t + T)}{[x_n(t) - x_{n+1}(t)]^\ell} [\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{7}$$

It can be seen that when m = 0 and ℓ = 0, the general equation becomes Eq. 3, while the condition m = 0 and ℓ = 1 converts the general equation to Eq. 5. Eq. 7 and the exponents m and ℓ will be shown to be significantly important in later portions of this paper.

Macroscopic theories of traffic flow date back to 1935. Greenshields (6), after inspection of a set of speed-density measurements, hypothesized that a linear relationship existed between speed and density:

$$u = u_f [1 - \frac{k}{k_j}] \tag{8}$$

where

u_f = free-flow speed,
k_j = jam density,
u = speed, and
k = density.

Based on the developments of Lighthill and Witham (7), Greenberg (8), in 1959, proposed a macroscopic flow model by using the analogy of the traffic-flow situation with the problem of one-dimensional fluid flow. By using the equation of continuity and the equation of motion, a relationship between speed and density was developed:

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{9}$$

$$\frac{du}{dt} = -c^2\left(\frac{1}{k}\right)\frac{\partial k}{\partial x} \tag{10}$$

$$u = c\,\ln\left(\frac{k_j}{k}\right) \tag{11}$$

where $c = u_0$ = speed at maximum flow.

In 1960, Underwood (9) proposed an exponential speed-density relation:

$$u = u_f e^{-k/k_0} \tag{12}$$

where k_0 = density at maximum flow.

Edie (10) after careful study of q - k curves, hypothesized that there were two regimes of traffic flow: free flow and congested flow. He proposed that an exponential speed-density relation be used for the free-flow regime, Eq. 12, and the Greenberg equation be used for the congested-flow regime, Eq. 11.

Using the fluid-flow analogy approach proposed earlier by Greenberg but employing a more general derivation of this problem, Drew (11) proposed the following speed-density relation:

$$\frac{du}{dt} = -c^2 k^n \frac{\partial k}{\partial x} \tag{13}$$

$$u = u_f\left[1 - \left(\frac{k}{k_j}\right)^{\frac{n+1}{2}}\right] \quad \text{for } n > -1 \tag{14}$$

Drake, May, and Schofer (12) report on the application of a bell-shaped curve which gave satisfactory results when compared to speed-density measurements:

$$u = u_f e^{-\frac{1}{2}\left(\frac{k}{k_0}\right)^2} \tag{15}$$

The speed-density relationships shown in Eqs. 8, 11, 12 as well as two linear regimes, three linear regimes, and a modified Greenberg model were evaluated in this study.

In 1961, a paper (5) of major significance was published that placed emphasis on the steady-state flow equations which result from various microscopic theories of traffic flow. It was shown that several proposed macroscopic theories are mathematically equivalent to the generalized microscopic expression given by Eq. 7, provided proper integers are selected for the exponents m and ℓ. For example, if m and ℓ are assumed to have the values of 0 and 2, respectively, the microscopic expression and Greenshields' macroscopic expression will give identical speed-density relationships. Thus this paper offered the first evidence of the bridge between microscopic and macroscopic approaches of traffic-flow theory.

Further analysis of the interrelationships between the two different approaches was given by Haight (13). Drew also investigated this interrelationship and showed that by setting m = 0 and varying the exponent, ℓ, Eq. 7 can be transformed into the steady-state flow Eq. 14, for $n = 2\ell - 3$ (11).

MATRIX DEVELOPMENT AND RELATIONSHIP OF MACROSCOPIC AND MICROSCOPIC THEORIES

The use of the general expression for the sensitivity factor in the stimulus-response equation (Eq. 6) as formulated by Gazis, Herman, and Rothery (5) gives a very powerful tool for an evaluation of existing models. All previously mentioned models can be described by the generalized equation (Eq. 7) by using appropriate m and ℓ values. Gazis et al have shown that by integration of the generalized equation the following expression is obtained:

$$f_m(u) = c' + cf_\ell(s) \tag{16}$$

where

u = steady-state speed of a stream of traffic,
s = constant average spacing, and
c and c′ = some appropriate constants consistent with physical restrictions.

The integration constant c′ is related to a free speed, u_f, or a jam spacing, s_j, depending on the values of m and ℓ. The jam spacing, s_j, can be transformed to jam density, k_j, by $s_j = 1/k_j$.

By using this general solution of Gazis et al, a matrix of steady-flow equations for different m and ℓ values was developed. The general expressions are shown in Figure 2, and the expressions in macroscopic model format are shown in Figure 3. An inspection of these two matrices reveals that all of the previously reported microscopic and macroscopic models and several other possible models can be located in terms of m and ℓ combinations. For example, the models of Pipes (2), Reuschel (1), and Chandler, Herman, and Montroll (3) are obtained when m = 0 and ℓ = 0. The Greenberg model (8) and the Gazis, Herman, and Potts model (4) is obtained when m = 0 and ℓ = 1. When m = 0 and ℓ = 3/2, the Drew model (11) is obtained. The Greenshields model (6) results when m = 0 and ℓ = 2, whereas the Edie (10) and Underwood (9) model results when m = 1 and ℓ = 2. The bell-shaped curve proposed by Drake, May, and Schofer (12) is obtained when m = 1 and ℓ = 3.

The matrix of m and ℓ values not only shows that the existing traffic-flow models can be reduced to the generalized car-following model, but also that, by choosing particular m and ℓ combinations, a wide variety of shaped curves for the speed-density relation can be selected (Fig. 4). One also can recognize certain trends in the shape of the curves by keeping one of the exponents, m or ℓ, constant. It should be noted that

ℓ \ m	m < 1	m = 1	m > 1
$\ell < 1$	$u^{1-m} = ck_j^{\ell-1} + ck^{\ell-1}$	Boundary conditions not satisfied	$u^{1-m} = u_f^{1-m} + ck^{\ell-1}$
$\ell = 1$	$u^{1-m} = c\ln(1/k_j) + c\ln(1/k)$	$\ln u = c\ln(1/k_j) + c\ln(1/k)$	$u^{1-m} = c\ln(1/k_j) + c\ln(1/k)$
$\ell > 1$	$u^{1-m} = ck_j^{\ell-1} + ck^{\ell-1}$	$\ln u = \ln u_f + ck^{\ell-1}$	$u^{1-m} = u_f^{1-m} + ck^{\ell-1}$

Figure 2. Matrix of steady-state flow equations for different m, ℓ values in $f_m(u) = c' + cf_\ell(s)$.

$\ell \backslash m$	m = 0	m = 1
$\ell = 0$	$u = \frac{1}{s}\left[\frac{1}{k} - \frac{1}{k_j}\right]$	-
$\ell = 1$	$u = u_0 \ell n \left(\frac{k_j}{k}\right)$	-
$\ell = 1.5$	$u = u_f\left[1 - \left(\frac{k}{k_j}\right)^{1/2}\right]$	-
$\ell = 2$	$u = u_f\left[1 - \left(\frac{k}{k_j}\right)\right]$	$u = u_f e^{-(k/k_0)}$
$\ell = 3$	-	$u = u_f e^{-\frac{1}{2}(k/k_0)^2}$

Figure 3. Matrix of existing traffic-flow models.

non-integer m and ℓ values can be utilized, and consequently an expression can be determined which more closely represents actual speed-density relations. This is shown in Figure 5, where for a constant exponent m = 1 the exponent ℓ is changed in steps of 2/10. One can see the gradual change from the exponential model (m = 1, ℓ = 2) to a bell-shaped model (m = 1, ℓ = 3).

In examining the matrix, one should remember that the m value is the exponent of the following vehicle's speed $[\dot{x}_{n+1}(t+T)]$ and the ℓ value is the exponent of the spacing of the two vehicles $[x_n(t) - x_{n+1}(t)]$. Consequently, the fundamental difference between models is the weight given to the following vehicle's speed and the spacing between vehicles.

This matrix of m and ℓ values has permitted the development of analytical techniques for evaluating deterministic traffic-flow models using speed-density measurements.

ANALYTICAL PROCEDURE FOR EVALUATING DETERMINISTIC INTEGER AND NON-INTEGER TRAFFIC-FLOW MODELS

The speed-density relation rather than the flow-density or speed-flow relation was selected as the relationship for evaluation. Once this equation is evaluated, the other relationships can be obtained by using the steady-state equation q = uk. The speed-

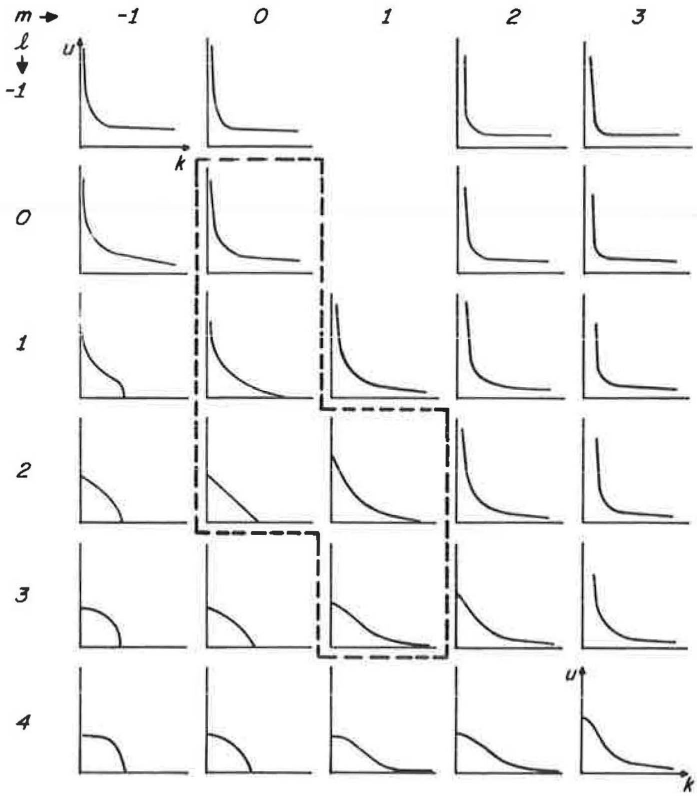

Figure 4. Matrix of speed-density relations for various m, ℓ combinations of the general car-following equation.

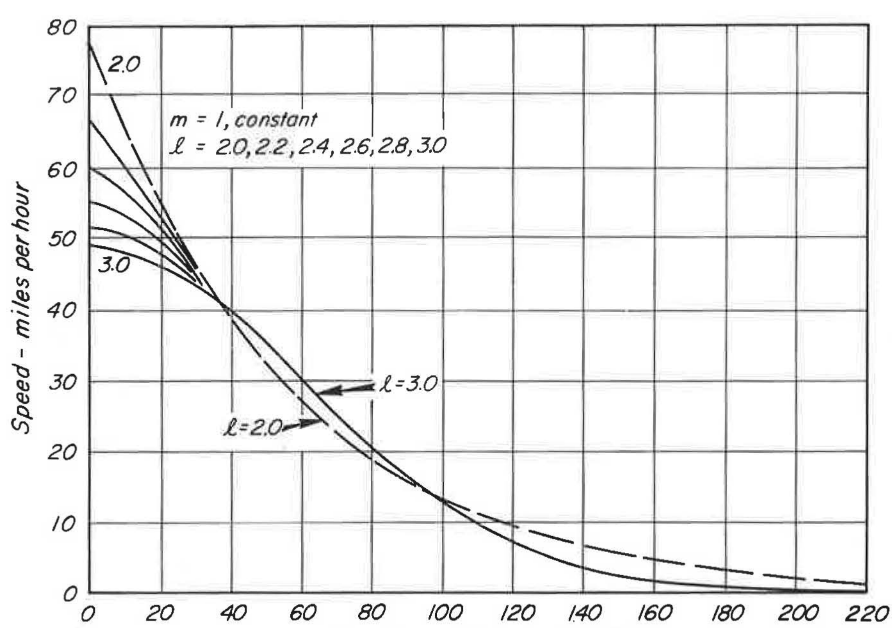

Figure 5. Influence of the use of non-integer exponents on the speed density relation.

density relation has the advantage of being easier to handle mathematically. For example, the equation resulting from the integration of Eq. 7 is in the form of a linear speed-density relationship depending on the scale of the coordinate system.

A variety of different models can be obtained through the selection of different m, ℓ values. To define a specific equation, the 4 parameters m, ℓ, c, and c$'$ must be determined. One method to obtain these parameters is the use of regression analysis or least squares fit to compute c and c$'$, and to determine the parameters m and ℓ by maximizing or minimizing the correlation coefficient or the standard error of estimate in cooperation with close approximation to significant traffic-flow characteristics.

Statistical Procedure of Evaluation

As shown in Figure 4, the speed-density relation is truly a linear equation with a regular coordinate system, when m = 0 and ℓ = 2. For the speed-density relationships with other exponents m and ℓ, the speed, u, and/or density, k, have to be transformed (depending on the parameters m and/or ℓ), if linear regression analysis is to be employed. This is equivalent to contracting or extending the scale on the ordinate and/or abscissa in order to arrange the data points for a linear analysis. The regression analysis is used to obtain an estimate of the dependent variable (in this case, speed, u) from an independent variable (in this case, density, k). The regression equation is determined by minimizing the squared deviations of the data points from the regression curve. However, the correlation coefficient obtained from the regression analysis for the transformed scale (where each speed-density relation is made linear) is not the same as for the real scale (where each speed-density relation except one is not linear).

One possible solution to the problem would be to compute a new correlation coefficient for the real-scale situation similar to the linear correlation coefficient, which was obtained from the transformed-scale situation. However, the linear regression and correlation analysis is based on the supposition that the mean values of the dependent

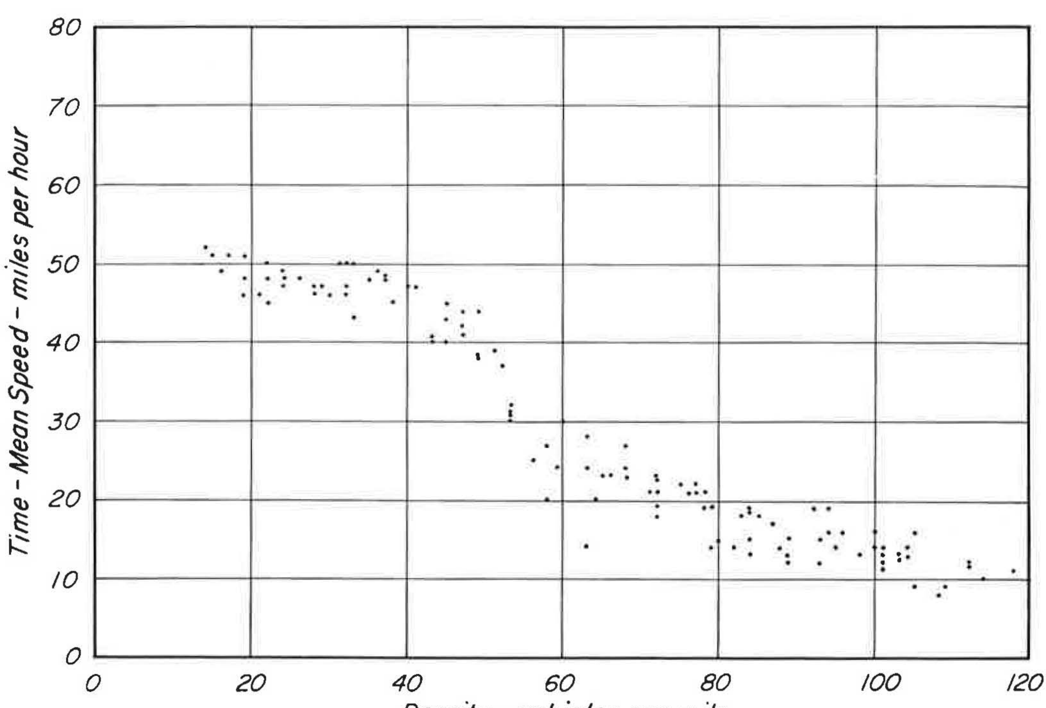

Figure 6. Speed-density data collected on the Eisenhower Expressway.

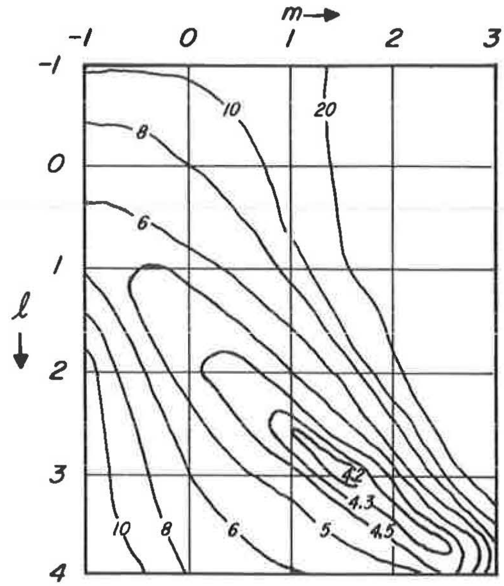

Figure 7. Curves of equal mean deviations (mpm).

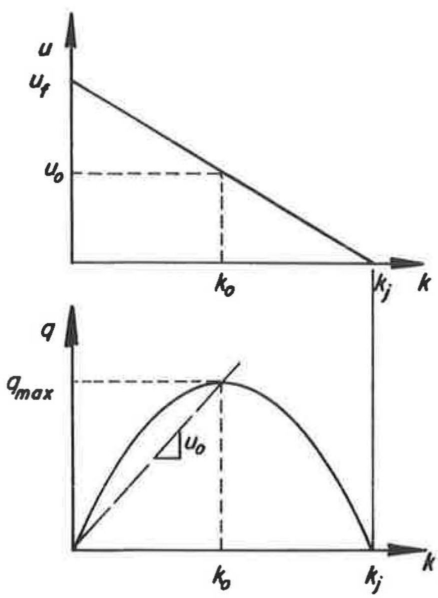

Figure 8. Traffic-flow characteristics shown for the linear speed-density model of Greenshields ($m = 0, \ell = 2$).

and independent variable are one point on the regression line. This is no longer the case for the retransformed regression curve in the real-scale system.

This and other reasons lead to the idea of computing the sum of the squared deviations of the data points from the regression curve for the dependent variable $\Sigma(u - u_{estimated})^2$ and taking this as a measurement of the goodness of fit. By dividing this sum by the number of data points, N, and taking the square root of this value, one obtains a mean deviation, s, for the curve considered.

This method was applied to a set of 118 one-minute samples of time-mean speeds and mean densities recorded with the pilot detection system of the Chicago Area Expressway Surveillance Project. The data were collected in the middle lane of the three-lane westbound roadway on the Eisenhower Expressway, at Harlem Avenue. The highest measured mean speed was 52.3 mph, the lowest 8.2 mph; the highest computed density was 118.4 vpm, the lowest 14.2 vpm (Fig. 6). A justification of using the time-mean speed instead of the space-mean speed with the corresponding density is given by Drake, May, and Schofer (12).

The results were plotted in an m, ℓ plane and values of equal mean deviation were combined to trace a contour map of equal deviations. Figure 7 shows this plot for the most significant values of the exponents m ($-1 \leq m \leq 3$) and ℓ ($-1 \leq \ell \leq 4$). The minimization of the mean deviation leads to an area of m and ℓ values, which gives a first indication about the goodness of fit of selected speed-density relations with particular m and ℓ combinations.

Traffic-Flow Characteristics as Criteria for Evaluation

The statistical procedure, minimizing of the mean deviations, alone is not satisfying. There is very little difference in the mean deviations for different sensitivity factors or m, ℓ combinations. On the basis of these small differences, it would be difficult to give the preference to certain speed-density relations as the best fit to the data with any assurance. The choice of a model or equation only by the criteria of the minimum mean deviation of the data points from the estimated curve may also give misleading results. The chosen equation may fit the data points very nicely, but the speed-density curve may, for example, have no limit value for the free speed. Physical restrictions and field observations, however, indicate that there are limit values for certain traffic-flow characteristics.

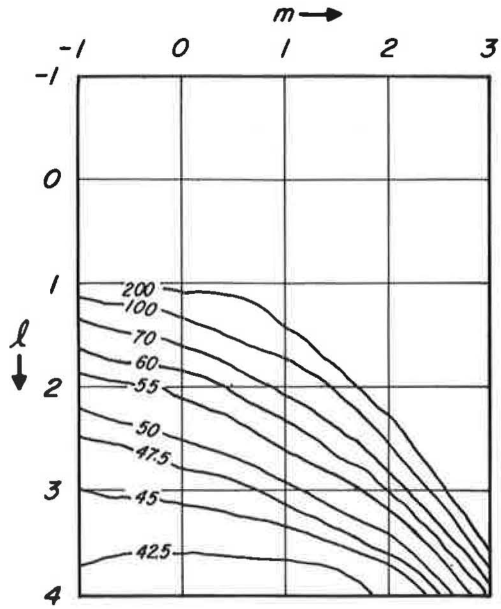

Figure 9. Curves of equal free speed (mph).

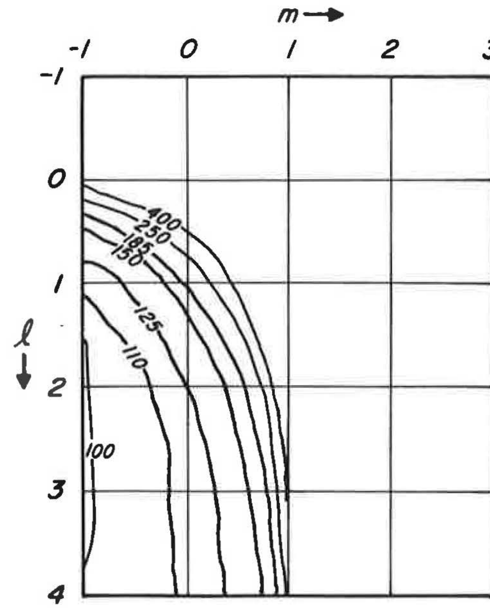

Figure 10. Curves of equal jam density (vpm).

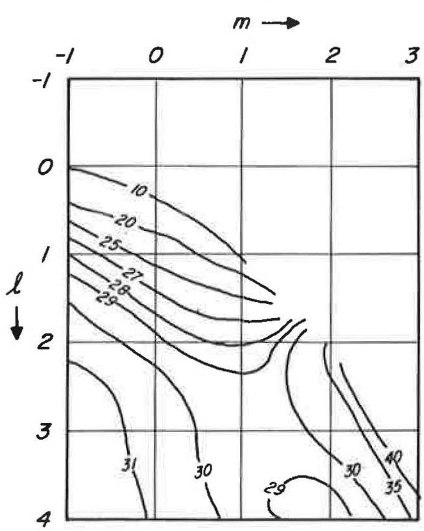

Figure 11. Curves of equal optimum speed (mph).

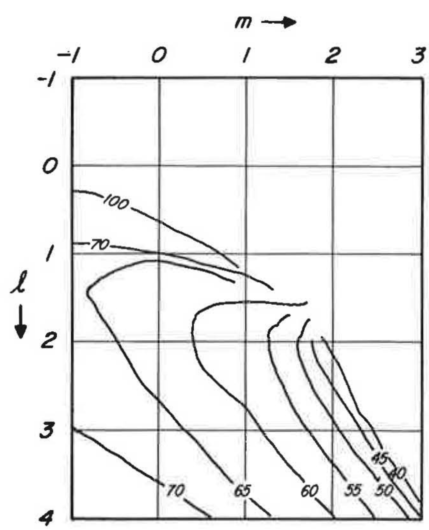

Figure 12. Curves of equal optimum density (vpm).

Important traffic-flow conditions for a speed-density relationship are the free speed, u_f, where the density $k = 0$ and the jam density, k_j, where the speed $u = 0$. Significant points of the flow-density relation are the optimum density, k_0, and the optimum speed, u_0, at which the maximum flow, q_{max}, occurs (Fig. 8). These flow characteristics can be derived from the steady-state flow equation (Eq. 16, see also Fig. 2) by setting the density $k = 0$ to obtain the free speed, u_f, and by setting the speed $u = 0$ to obtain the jam density, k_j. By differentiating the flow-density equation $q = uk$ with respect to the density, k, or the speed, u, and setting the results equal to zero $\partial q/\partial k = 0$ or $\partial q/\partial u = 0$, one

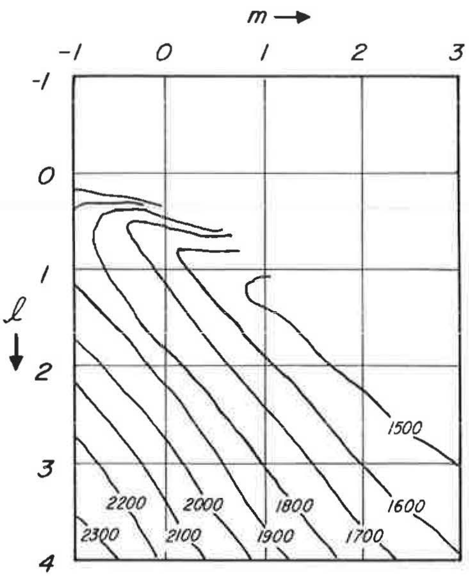

Figure 13. Curves of equal maximum flow (vph).

can obtain the optimum density, k_0, and the optimum speed, u_0. The flow-density relation $q = uk$ gives for $k = k_0$ and $u = u_0$ the maximum flow, q_{max}.

These traffic-flow characteristics, which can be determined for each m, ℓ combination, allow a judgment about the value of the specific model considered and provide a good means of evaluation. For the evaluation, it is helpful to plot the results in an m, ℓ plane and to develop curves of equal levels of flow characteristics. The trends of these curves show how specific m, ℓ combinations or models fit the flow requirements. The model which fits the flow characteristics best can be determined by superimposing the curves of the flow characteristics graphically and by locating in this way that area of m, ℓ combinations which fulfills all or most flow requirements as well as statistical measures of deviation.

This procedure, the use of the flow characteristics as a means of evaluation, was applied to the previously mentioned data from the Eisenhower Expressway. Figures 9 and 10 show the results of the computations for free speed, u_f, and jam density, k_j. The pictures show clearly that finite values for the free speed exist only for $\ell > 1$ and for the jam density only for $m < 1$. The calculations for the other areas give either unrealistic results or, for example, infinite values for the jam density when $m \geq 1$. The curves of equal free speed and/or jam density indicate that only very small bands give reasonable flow characteristics in the m, ℓ plane.

The investigation of the significant points of the flow-density equation is shown in Figures 11, 12, and 13. Realistic values for the optimum density, the optimum speed, and the maximum flow occur only in a limited area of the m, ℓ plane. This area

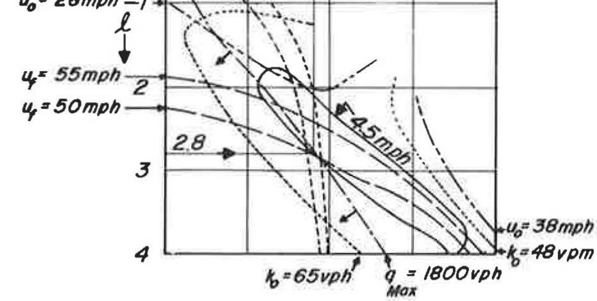

Figure 14. Superposition of evaluation criteria.

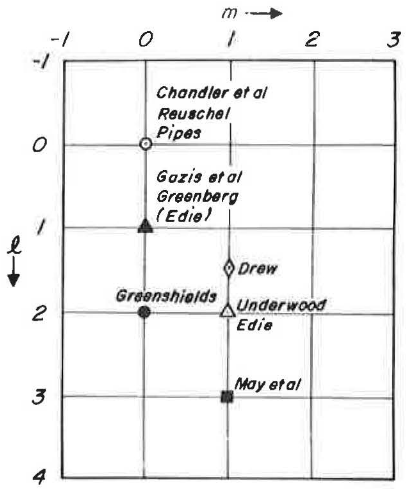

Figure 15. Existing traffic-flow models.

is the part below the bisecting diagonal through the origin of the coordinate system. The curves of equal optimum speed and optimum density having reasonable values cover a relatively wide range of the m, ℓ plane, whereas the curves of equal maximum flow show a trend of increasing flow parallel to the bisecting diagonal.

While the previously mentioned contour maps of traffic-flow characteristics can be used individually to check the applicability of a certain model, the graphical superposition of all contour maps gives a tool to determine those in m, ℓ combinations, which fit best the assumed flow requirements. Figure 14 shows the superposition of these parameters for certain reasonable values. According to the situation at the Eisenhower Expressway, the following values for the flow characteristics have been assumed to be reasonable: (a) free speed, u_f = 50-55 mph; (b) jam density, k_j = 185-250 vpm; (c) optimum speed, u_0 = 28-38 mph; (d) optimum density, k_0 = 48-65 vpm; and (e) maximum flow, q_{max} ≥ 1,800 vph. The strongest requirements are imposed by the free speed and the jam density shown as narrow bands on the m, ℓ chart (Fig. 14). The other limitation is given through the requirement of a greater maximum flow rate than 1,800 vehicles per hour. While the optimum density, optimum speed, and mean deviation are of less influence, these three criteria limit the solution to a very small area around m = 0.8 and ℓ = 2.8, where all traffic-flow criteria are fulfilled.

	Selected model for integer solution	Selected model for non-integer solution
m	1.0	0.8
ℓ	3.0	2.8
mean deviation	4.6 mph	4.5 mph
free speed	48.7 mph	50.1 mph
jam density	∞	220 vpm
optimum speed	29.5 mph	29.6 mph
optimum density	60.8 vpm	61.1 vpm
maximum flow	1,795 mph	1,810 mph
macroscopic equation	$u = u_f e^{-\frac{1}{2}\left(\frac{k}{k_0}\right)^2}$	$u = u_f \left[1 - \left(\frac{k}{k_j}\right)^{1.8}\right]^5$
microscopic equation	$\ddot{x}_{n+1}(t + T) = (a) \frac{\dot{x}_{n+1}(t + T)^1}{[x_n(t) - x_{n+1}(t)]^3} [\dot{x}_n(t) - \dot{x}_{n+1}(t)]$	$\ddot{x}_{n+1}(t + T) = (a) \frac{\dot{x}_{n+1}(t + T)^{0.8}}{[x_n(t) - x_{n+1}(t)]^{2.8}} [\dot{x}_n(t) - \dot{x}_{n+1}(t)]$
(a) value	1.35×10^{-4}	1.33×10^{-4}

Figure 16. Comparison of the results for the selected integer and non-integer traffic-flow models.

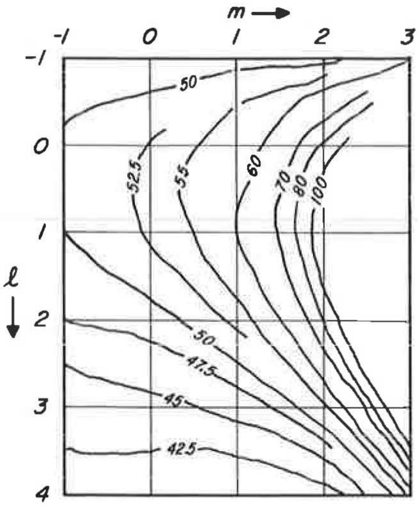

Figure 17. Curves of equal speed (mph) at density k = 20 vpm.

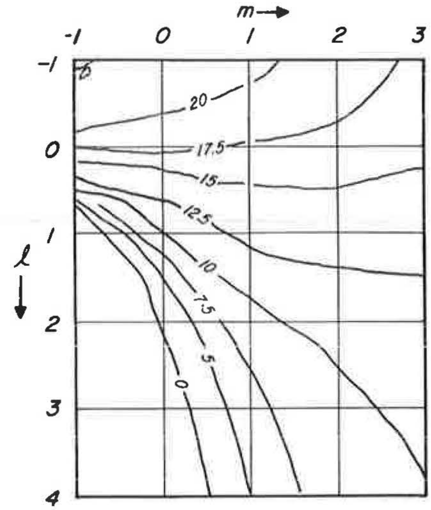

Figure 18. Curves of equal speed (mph) at density k = 120 vpm.

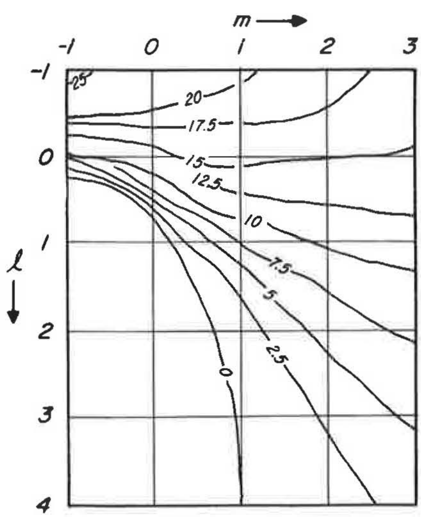

Figure 19. Curves of equal speed (mph) at density k = 220 vpm.

The superposition of the evaluation criteria in Figure 14 shows that the use of a continuum of non-integer m, ℓ combinations allows a very good adjustment to the different levels of the evaluation requirements. The speed-density equation is

$$u = u_f[1 - (\frac{k}{k_j})^{1.8}]^5$$

where
u_f = 50.1 mph, and
k_j = 220 vpm.

Figure 15 gives, for reasons of comparison, the location of existing traffic-flow models in the m, ℓ plane.

It can be seen that of the existing models tested with the data set, the bell-shaped curve for m = 1, ℓ = 3 shows the best fit to the criteria. A comparison of the selected model for the integer and non-integer m, ℓ combination is made in Figure 16. The advantage of the selected non-integer model over the integer model for the tested data set is that it has a finite jam density. It also has a slightly higher free speed and maximum flow.

As shown before, the free speed and the jam density require a very strong limitation in the application of certain models. An investigation of the speeds near these limit values has been undertaken to extend the procedure of evaluation (Figs. 17, 18, 19). Figure 17 shows that the area limited by the levels of acceptable free speed in Figure 14 can be considerably extended, if one takes the speed at low densities, for example, k = 20 vpm as criterium. A similar effect can be shown for speeds at densities of k = 120 vpm (Fig. 18). The investigation of speeds which occur at the density close to the assumed jam density, here 220 vpm, shows the contour map in Figure 19. If one allows a speed of 2 or 3 mph at this assumed density, the area limited by the jam density levels in Figure 14 can be extended to larger values of m (2.0 to 2.5). This allows the inclusion of more models, which still fulfill all flow characteristic criteria requirements, except for the desired maximum flow rate. But, as has been mentioned before, the maximum flow is an important criterium, and it cannot be neglected.

The superposition of the criteria of the flow characteristics with that of the statistical analysis is also shown in Figure 14. A mean deviation of 4.5 mph was considered as the limit for the evaluation procedure. The graph shows that the minimization of the mean deviation already gave a good means of evaluation. This is especially true if the extended criteria shown in Figures 16 and 18 are included. The area indicated by the statistical criterium coincides with all of the criteria for the flow characteristics. This stresses that already very small differences in the mean deviations investigated are of great significance. The only criterium which is not included is the maximum flow rate of more than 1,800 vph. The curve of 1,800 vph is just at the border of the 4.5-mph mean deviation curve. This justifies the use of the flow characteristics criteria as a means of evaluation, because only in this way is attention given to the desired and reasonable shape of the speed-density relationship.

SUMMARY

Study Results

This paper describes a procedure on how deterministic microscopic and macroscopic traffic-flow models can be evaluated. It has been shown that all reported macroscopic and microscopic traffic-flow models can be reduced to the general car-following equation (Eq. 7) by selection of appropriate exponents m and ℓ, representing the influence of the speed of the following vehicle, respecitvely, of the distance headway between vehicles on the sensitivity factor. The use of an m, ℓ matrix gives the possibility of comparison of existing models due to given criteria. The m, ℓ plane is the basis for the method of evaluation applied. Two different procedures have been used. The statistical analysis is based on the minimizing of the mean deviations of the data points from the determined regression curve. The preference to this method was given because of its clearness. Although there are very little differences in the mean deviations for particular m, ℓ combinations, this method has been shown to be very effective. The other procedure introduces the traffic-flow characteristics as evaluation criteria. This provides a very helpful tool, because certain physical restrictions or limitations of the flow characteristics are considered as a control. The graphical superposition of the results of both evaluation procedures allows a judgment about the goodness of fit of existing traffic-flow models to given criteria and an estimate of those m, ℓ combinations (or models) which best fit the investigated data. The introduction of a continuum of non-integer exponents m and ℓ implies a considerably greater variety of possible models and a more flexible adjustment to the assumed criteria of evaluation.

The combination of m = 0.8 and ℓ = 2.8 fulfills all requirements of the assumed evaluation criteria, mean deviation, free speed, jam density, optimum speed, optimum density, and maximum flow. The evaluation indicates that the area around the line between m = 0.5, ℓ = 2.5 and m = 2.5, ℓ = 3.5 covers models of very good fit, but with a maximum flow rate of less than 1,800 vph. Models with a maximum flow rate greater than 1,800 vph appear in the area below this line. As can be seen from Figure 4, the speed-density relation tends to be bell-shaped in that area.

The investigation of the flow-density relation in these optimum combinations of m, ℓ values indicates a unique shape of the flow-density curve, i.e., in the high-density regime where the relationship exhibits a reversed curve.

The analysis shows that the general car-following equation implies a series of models in addition to known existing models, which describe actual data quite well. An extension to the area of existing models in the m, ℓ matrix can be proposed mainly for ($1 \le m \le 2.5$) and ($2.5 \le \ell \le 4.0$).

Future Directions

The use of electronic computers facilitates the procedure of evaluation considerably and an evaluation program has been written in Fortran language. The application of a plotter would have been helpful for the tracing of the contour maps.

The use of the developed evaluation procedure opens a wide field for the application of this method to other typical data sets. The evaluation of data from highways, tunnels, or urban streets may result in different m, ℓ combinations. In this way one can determine areas in the m, ℓ plane which are characteristic for certain types of traffic facilities.

The study should also be extended to the evaluation of multi-regime systems. Several authors have mentioned that these give a better description of actual data for the speed-density relationship.

REFERENCES

1. Reuschel, A. Fahrzeugbewegungen in der Kolonne. Oesterreichisches Ingenieurarchiv, Vol. 4, pp. 193-215, 1950. (Also Zeitschrift d. Oesterr. Ing. u. Arch. Vereins, Vol. 95, pp. 59-62, 73-77, 1950.)
2. Pipes, L. A. An Operational Analysis of Traffic Dynamics. Jour. Applied Physics, Vol. 24, pp. 274-281, 1953.
3. Chandler, R. E., Herman, R., and Montroll, E. Traffic Dynamics-Studies in Car-Following. Operations Research, Vol. 7, pp. 165-184, 1958.
4. Gazis, D. C., Herman, R., and Potts, R. Car-Following Theory of Steady-State Traffic Flow. Operations Research, Vol. 7, pp. 499-595, 1959.
5. Gazis, D. C., Herman, R., and Rothery, R. W. Nonlinear Follow-the-Leader Models of Traffic Flow. Operations Research, Vol. 9, pp. 545-567, 1961.
6. Greenshields, B. D. A Study in Highway Capacity. HRB Proc., Vol. 14, p. 468, 1934.
7. Lighthill, M. T., and Witham, G. B. On Kinematic Waves, A Theory of Traffic Flow on Long Crowded Roads. Proc. Royal Society, London, Series A, Vol. 229, 1955.
8. Greenberg, H. An Analysis of Traffic Flow. Operations Research, Vol. 7, pp. 79-85, 1959.
9. Underwood, R. T. Speed, Volume, and Density Relationships. Quality and Theory of Traffic Flow. Yale Bureau of Highway Traffic, pp. 141-188, 1961.
10. Edie, L. C. Car Following and Steady-State Theory for Non-Congested Traffic. Operations Research, Vol. 9, pp. 66-76, 1961.
11. Drew, D. R. Deterministic Aspects of Freeway Operations and Control. Texas Transportation Institute, Research Rept. 24-4, June 1965.
12. Drake, T. S., Schofer, J. L., and May, A. D., Jr. A Statistical Analysis of Speed Density Hypotheses. Expressway Surveillance Project, Chicago, Rept. 16, May 1965.
13. Haight, F. A. Mathematical Theories of Traffic Flow. Academic Press, New York, 1963.