Surface Displacement of an Elastic Layer Under Uniformly Distributed Loads

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The surface displacement of an elastic layer on a rigid base (a soil-rock system) under uniformly loaded areas of various shapes is evaluated, and the displacement of a two-layer elastic system, where the upper layer is more compressible than the lower, under a uniformly loaded circular area is computed according to a rigorous solution of the theory of elasticity.

•THE stresses and displacements of a semi-infinite uniform elastic layer under a surface load was solved by Boussinesq (1). A solution for an elastic layer resting on a rigid base was initiated by Filon (2). Solutions for this system in several different cases were published by Melan (3), Marguerre (4), Biot (5), Pickett (6) and others during the thirty years after Filon's work. All of these works concerned the stresses on the base of this system.

On the other hand, Steinbrenner (7) introduced an approximate equation of the surface displacement under a rectangular loaded area of an elastic layer on a rigid base. His equation has been widely used to estimate the elastic displacement of a soil-rock system, because not enough was known about rigorous solutions of the surface displacement of this system.

One of the most important works in this field was that by Burmister (8). He evaluated stresses at several depths in an elastic layer and also surface displacement assuming Poisson's ratios of 0.2 and 0.4. Recently, Mandel (9), Egorov (10) and Sovinc (11) contributed in some evaluations connected with displacement of this system.

However, not enough solutions of surface displacements of this system are available to determine the accuracy of Steinbrenner's equation. Thus, the present authors evaluated the surface displacement of the system under uniformly loaded areas of various shapes according to a rigorous solution of the theory of elasticity. Also, they computed the surface displacement of a two-layer elastic system, where the upper layer is more compressible than the lower, under a uniformly loaded circular area, and they compared this solution with the previous case.

SURFACE DISPLACEMENT OF AN ELASTIC LAYER ON A RIGID BASE

Evaluated Items for an Elastic Layer on a Rigid Base

In general, the vertical surface displacement of an elastic layer on a rigid base, w, in expressed as follows:

$$w = \frac{p}{E} B I \tag{1}$$

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where

p = uniformly distributed pressure on loaded area,
B = width of loaded area,
E = modulus of elasticity of upper layer, and
I = surface displacement influence value which is a function of Poisson's ratio of the upper layer, ratio of thickness T of upper layer to width of loaded area, shape of loaded area, and condition at the interface.

The displacement influence values, I, approach limit values at thickness T = ∞ as follows:

For the center of a loaded circular area:

$$I_{co} = 2(1-\mu^2) \qquad (2)$$

where

I_{co} = $w_{co} / \frac{p}{E} a$
w_{co} = the displacement at the center of the loaded circular area, and
a = radius of the loaded circular area.

For the corner of a loaded rectangular area:

$$I_{rc} = \frac{(1-\mu^2)}{\pi}\left[\lambda \log_e \frac{1+\sqrt{\lambda^2+1}}{\lambda} + \log_e\left(\lambda+\sqrt{\lambda^2+1}\right)\right] \qquad (3)$$

where

I_{rc} = $w_{rc} / \frac{p}{E} B$,
w_{rc} = the displacement at the corner of the loaded rectangular area,
B = width of the loaded rectangular area,
λ = L/B, and
L = length of the loaded rectangular area.

Uniformly Loaded Circular Area

The displacement influence value, I_{co}, for the center of a uniformly loaded circular area was computed in the cases of an adhesive interface and a smooth interface between the elastic layer and the rigid base with the same procedure as explained in another paper by the present authors (12). This influence value may be expressed as follows:

$$I_{co} = w_{co} / \frac{p}{E} a = f(\delta_a, \mu, \text{assumption at interface}) \qquad (4)$$

where

w_{co} = the displacement at the center of the loaded circular area (Fig. 1),
a = the radius of the loaded circular area, and
δ_a = T/a = the thickness factor for the loaded circular area.

Other symbols are as explained before.

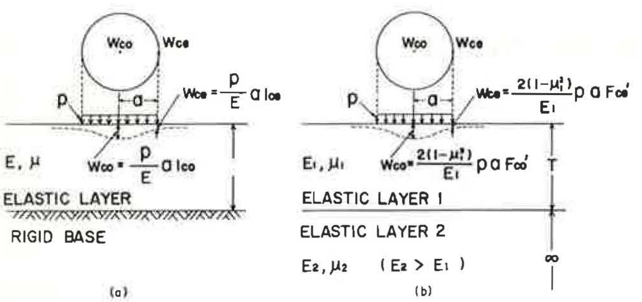

Figure 1. Displacement of an elastic layer under a uniformly loaded circular area: (a) elastic layer on a rigid base; (b) elastic layer on a stiffer layer.

Figure 2 shows the relations between the displacement influence value and the thickness factor for μ = 0, 0.1, 0.2, 0.3,

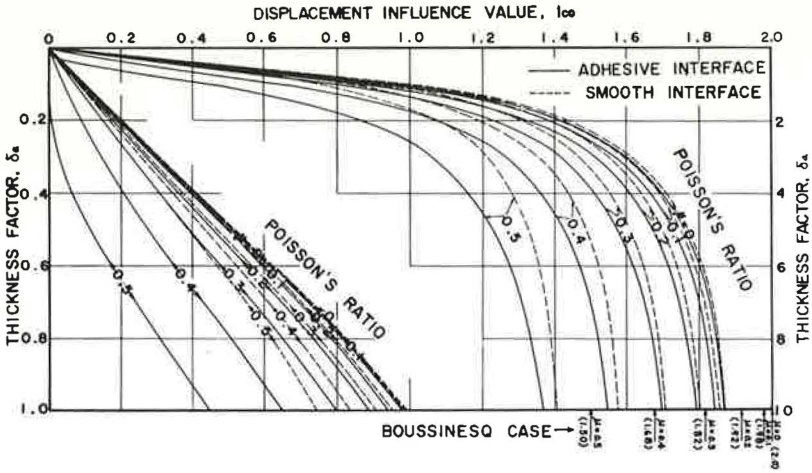

Figure 2. Relation between displacement influence value, I_{co}, and thickness factor, δ_a, at the center of a loaded circular area on an elastic layer on a rigid base.

0.4, and 0.5. These curves approach the values of Boussinesq case, i.e., $I_{co} = 2(1-\mu^2)$, as the thickness factor becomes higher.

Uniformly Loaded Strip Area

The displacement influence value, I_{so}, for the center of a uniformly loaded strip area, defined below for this case, was computed with the LGP-30 electronic computer according to the theory of elasticity in the cases of an adhesive interface and a smooth interface.

$$I_{so} = w_{so} \Big/ \frac{p}{E} b = f(\delta_b, \mu, \text{assumption at interface}) \tag{5}$$

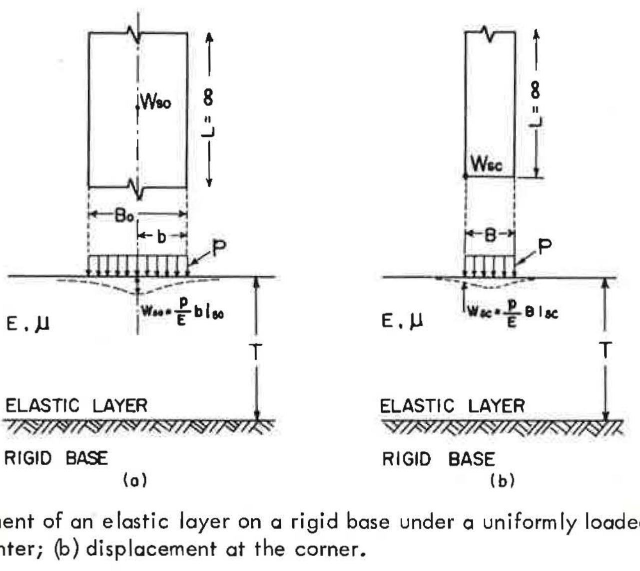

Figure 3. Displacement of an elastic layer on a rigid base under a uniformly loaded strip area: (a) displacement at the center; (b) displacement at the corner.

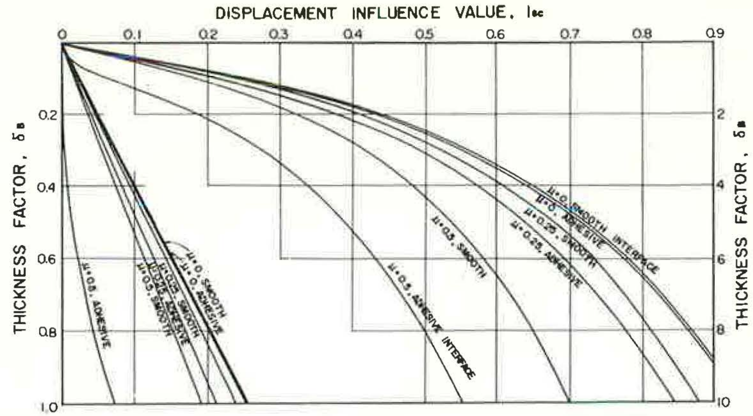

Figure 4. Relation between displacement influence value, I_{sc}, and thickness factor, δ_B, for the corner of a loaded strip area on an elastic layer on a rigid base.

where

w_{so} = the displacement at the center of the loaded strip area (Fig. 3a), and
δ_b = T/b = the thickness factor for the center of the loaded strip area.

From these computed results, the displacement influence value, I_{sc}, for the corner of a uniformly loaded strip area was derived based on the principle of superposition.

$$I_{sc} = w_{sc} \Big/ \frac{p}{E} B = f(\delta_B, \mu, \text{assumption at interface}) \tag{6}$$

where

w_{sc} = the displacement at the corner of the loaded strip area (Fig. 3b), and
δ_B = T/B = the thickness factor.

The computed results for I_{sc} are shown in Figure 4, which shows that Poisson's ratio and the condition of interface have remarkable effect on the displacement influence value around $\mu = 0.5$, but small effect on the value around $\mu = 0$.

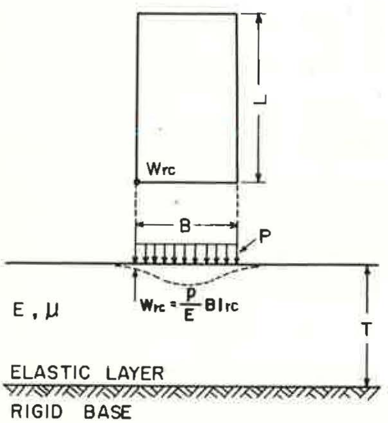

Figure 5. Displacement of an elastic layer on a rigid base under a uniformly loaded rectangular area.

Uniformly Loaded Rectangular Area

The displacement influence value, I_{rc}, for the corner of a uniformly loaded rectangular area was evaluated graphically based on the results for the center of a uniformly loaded circular area.

$$I_{rc} = w_{rc} \Big/ \frac{p}{E} B = f(\delta_B, \lambda, \mu, \text{assumption at interface}) \tag{7}$$

where

w_{rc} = the displacement at the corner of the loaded rectangular area (Fig. 5),
λ = L/B = the length factor, and
L = the length of the loaded rectangular area.

The results in the cases of $\mu = 0.5$ and $\mu = 0$ are computed. Figures 6 through 11 for $\mu = 0.5$, 0.4, 0.3, 0.2, 0.1, and 0 were based on Burmister's

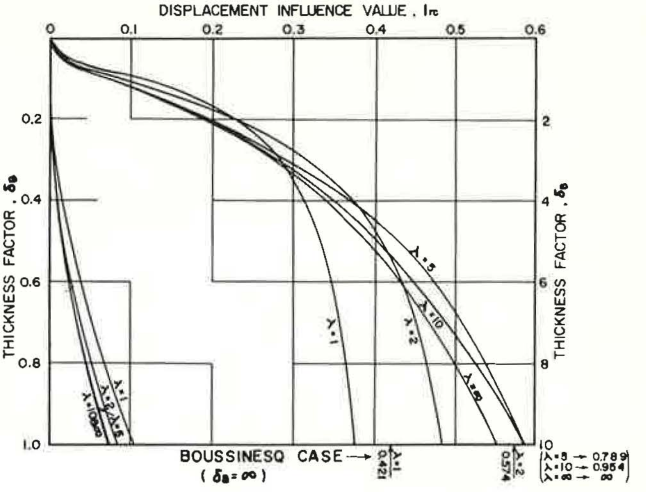

Figure 6. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0.5, adhesive interface).

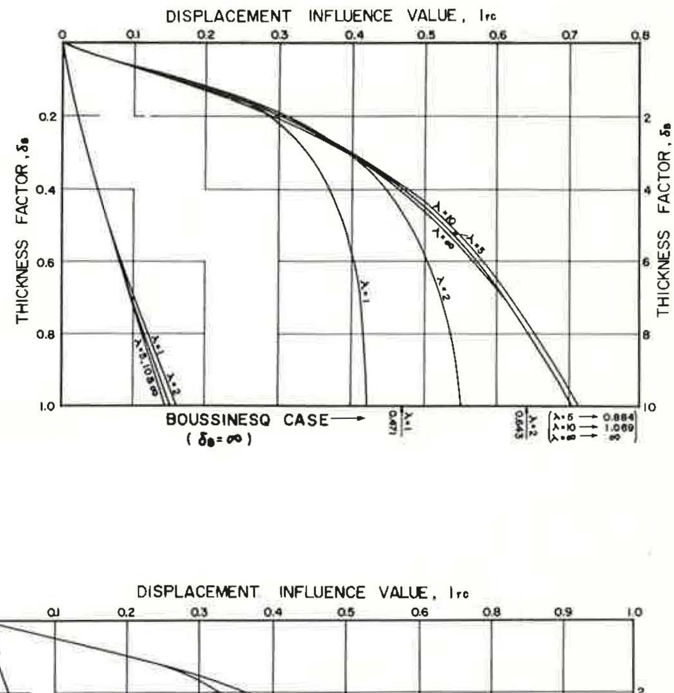

Figure 7. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0.4, adhesive interface).

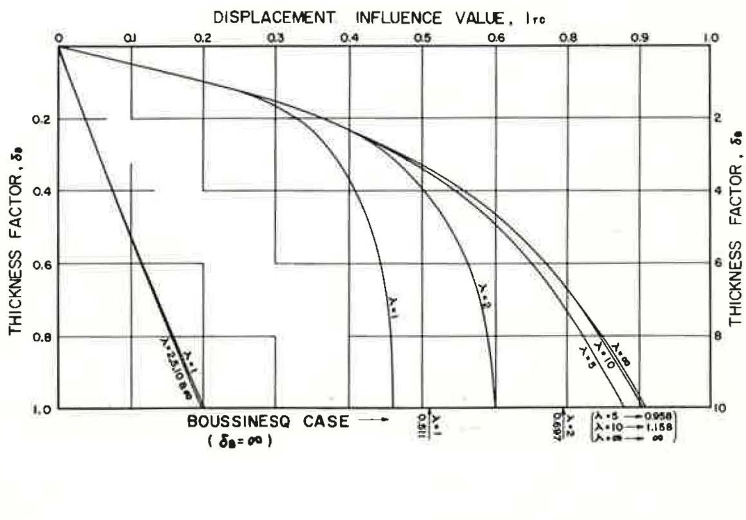

Figure 8. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0.3, adhesive interface).

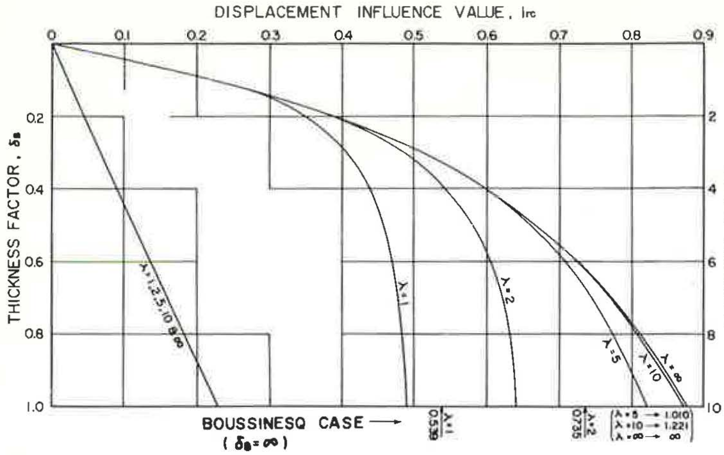

Figure 9. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0.2, adhesive interface).

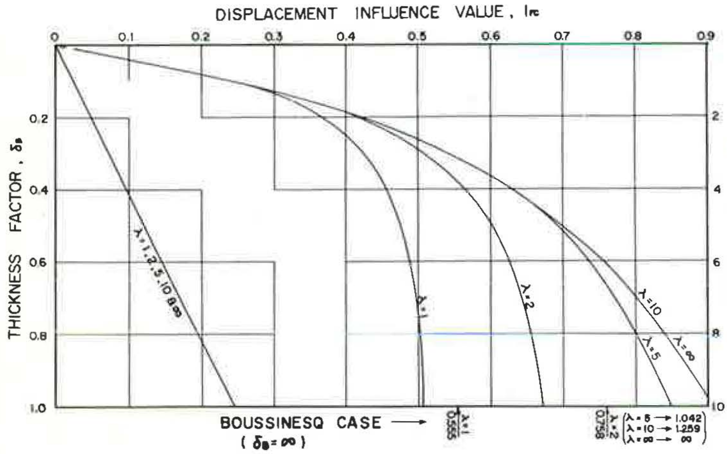

Figure 10. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0.1, adhesive interface).

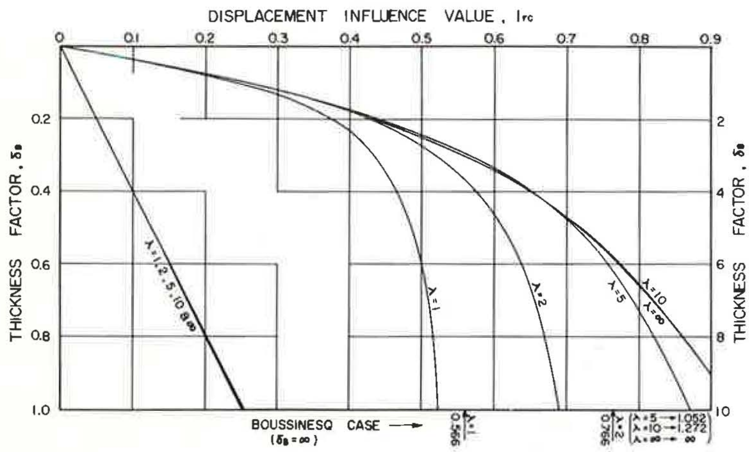

Figure 11. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base (μ = 0, adhesive interface).

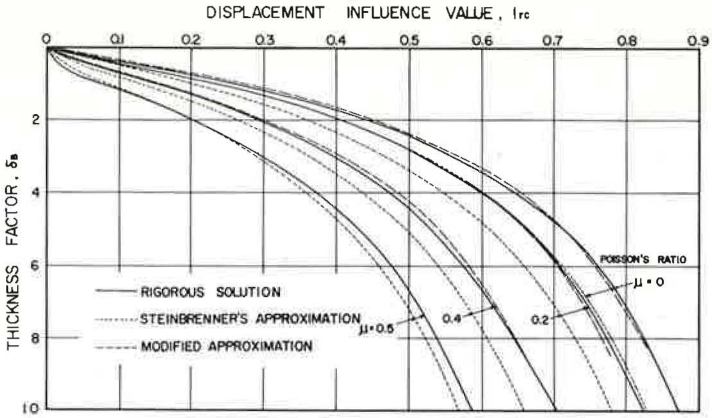

Figure 12. Relation between displacement influence value, I_{rc}, and thickness factor, δ_B, for the corner of a loaded rectangular area on an elastic layer on a rigid base, according to the rigorous solution of elasticity, Steinbrenner's approximation and a modified approximation (λ = 5, adhesive interface).

computation (8), authors' computation and a graphical interpolation. Based on the principle of superposition the displacement influence values, I_{ro} for the center and I_{re} for the middle of edge of a loaded rectangular area can readily be determined.

Steinbrenner (7) proposed an approximate equation to estimate the displacement influence value, I_{rc}, for the corner of a loaded rectangular area of an elastic layer on a rigid base as follows:

$$I_{rc} = (1-\mu^2)\, I_1 + (1-\mu-2\mu^2)\, I_2 \tag{8}$$

where

$$I_1 = \frac{1}{\pi}\left[\lambda \log_e \frac{\left(1+\sqrt{\lambda^2+1}\right)\sqrt{\lambda^2+\delta_B^2}}{\lambda\left(1+\sqrt{\lambda^2+\delta_B^2+1}\right)} + \log_e \frac{\left(\lambda+\sqrt{\lambda^2+1}\right)\sqrt{1+\delta_B^2}}{\lambda+\sqrt{\lambda^2+\delta_B^2+1}}\right]$$

$$I_2 = \frac{\delta_B}{2\pi}\tan^{-1}\frac{\lambda}{\delta_B\sqrt{\lambda^2+\delta_B^2+1}}$$

This approximation is fairly good for μ = 0.5 but a little smaller for other Poisson's ratios (Fig. 12 shows the case of λ = 5, for example). To estimate more accurately the rigorous values with the Steinbrenner approximation the equivalent thickness factor δ_B' instead of δ_B may be used.

$$\delta_B' = n\delta_B \tag{9}$$

Where n = equivalent coefficient = 1.2 for μ = 0 to 0.4. These approximations are also shown in Figure 7.

SURFACE DISPLACEMENT OF AN ELASTIC LAYER ON A STIFFER LAYER

Although the displacement influence value was defined and used for an elastic layer on a rigid base, the displacement factor, F, is customarily expressed for a two-layer elastic system by

$$F_{co} = w_{co} \Big/ \left[\frac{2(1-\mu_2^2)}{E_2}\, pa\right] \tag{10}$$

$$F_{ce} = w_{ce} \Big/ \left[\frac{2(1-\mu_2^2)}{E_2}\, pa\right] \tag{11}$$

where

F_{co} = the displacement factor for the center of a uniformly loaded circular area on a two-layer elastic system,

F_{ce} = the displacement factor for the edge of a uniformly loaded circular area on a two-layer elastic system,

E_2 = the modulus of elasticity of the lower layer of the system, and

μ_2 = Poisson's ratio of the lower layer of the system.

Other symbols are as explained before.

Concerning a two-layer elastic system, Burmister (13) computed the displacement factor of the system where the upper layer is stiffer than the lower layer, and the displacement coefficient of an elastic layer resting on a rigid base was evaluated extensively by both Burmister (8) and the present authors. But, there are no theoretical data for the displacement factor of an elastic layer on a stiffer elastic layer, except Kirk's computation (14) of the factor for the center of a loaded circular area in the special case of $E_1/E_2 = 0.2$. Therefore, the displacement factor, F_{co} and F_{ce}, for the center and the edge of a loaded circular area on a two-layer system were computed for the cases where $E_1/E_2 = 0.01$, 0.1, 0.2, and 0.5, assuming $\mu_1 = \mu_2 = 0.5$.

To compare these results with the case of an elastic layer on a rigid base, it is convenient to use a modified displacement factor defined as follows:

$$F_{co}' = w_{co} \Big/ \left[\frac{2(1-\mu_1^2)}{E_1} pa \right] = \frac{E_1(1-\mu_2^2)}{E_2(1-\mu_1^2)} F_{co} \tag{12}$$

$$F_{ce}' = w_{ce} \Big/ \left[\frac{2(1-\mu_1^2)}{E_1} pa \right] = \frac{E_1(1-\mu_2^2)}{E_2(1-\mu_1^2)} F_{ce} \tag{13}$$

The modified displacement factors, F_{co}' and F_{ce}', for these cases are shown in Figures 13 and 14, compared with the Boussinesq case and the rigid base case.

Approximate equations of the modified displacement factors, F_{co}' and F_{ce}', were found based on the factors of the rigid base case.

$$F_{co}' = F_{cor}' + \frac{E_1}{E_2} (1-F_{cor}') \tag{14}$$

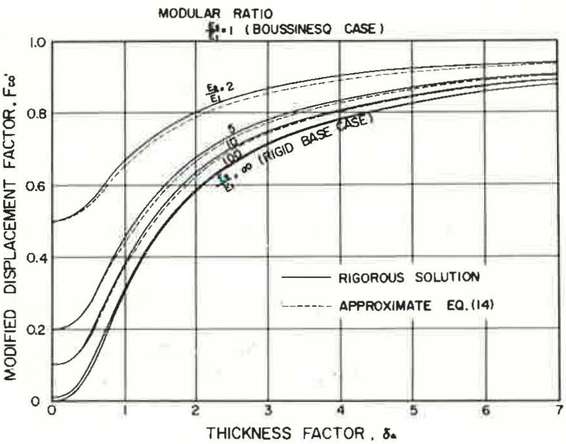

Figure 13. Relation between modified displacement factor, F_{co}', and thickness factor, δ_a, for the center of a loaded circular area on an elastic layer on a stiffer layer ($\mu_1 = \mu_2 = 0.5$, adhesive interface).

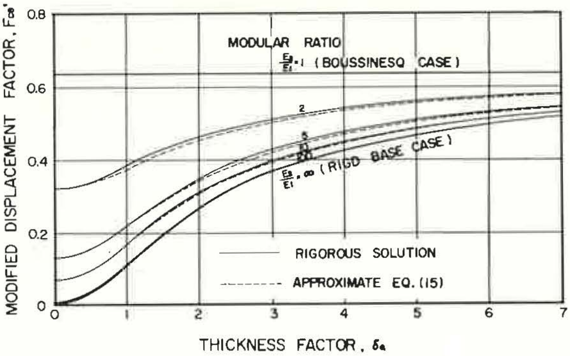

Figure 14. Relation between modified displacement factor, F_{ce}', and thickness factor, δ_a, for the edge of a loaded circular area on an elastic layer on a stiffer layer ($\mu_1 = \mu_2 = 0.5$, adhesive interface).

$$F_{ce}' = F_{cer}' + \frac{E_1}{E_2}(0.637 - F_{cer}') \tag{15}$$

where

$$F_{cor}' = \frac{1}{2(1-\mu_1^2)} I_{co}$$

= the modified displacement factor for the center of a uniformly loaded circular area on an elastic layer on a rigid base, and

$$F_{cer}' = \frac{1}{2(1-\mu_1^2)} I_{ce}$$

= the modified displacement factor for the edge of a uniformly loaded circular area on an elastic layer on a rigid base.

The factors calculated with Eqs. 14 and 15 are shown to compare with rigorous solutions in Figures 13 and 14.

CONCLUSIONS

1. The displacement of an elastic layer on a rigid base under a loaded circular area, under a loaded strip area or under a loaded rectangular area for any Poisson's ratio of the layer was evaluated in the form of the displacement influence value, based on the theory of elasticity.

2. Comparing Steinbrenner's approximation with these rigorous solutions, it was shown that Steinbrenner's method gives smaller displacements than the rigorous solutions, except for the case of a thin upper layer of $\mu \approx 0.5$. Although this approximation gives good estimation for the case of $\mu \approx 0.5$, for the case of $\mu = 0$ to 0.4 more reasonable approximations can be made by the use of an equivalent thickness factor, $\delta_B' = 1.2\,\delta_B$, in Steinbrenner's approximation.

3. Poisson's ratio and the condition of the interface have a remarkable effect on the displacement of an elastic layer on a rigid base when Poisson's ratio of the elastic layer approaches 0.5, but they have only a small effect on the displacement when Poisson's ratio tends to zero.

4. The displacement of an elastic layer on a stiffer layer under a loaded circular area was computed in the form of the displacement factor or the modified displacement factor, based on the theory of elasticity assuming Poisson's ratio of 0.5 for each layer. Approximate equations were proposed for the modified displacement factors of this kind of two-layer system from the displacement influence values of an elastic layer on a rigid base.

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