Theoretical Stress Distribution in an Elastic Multi-Layered System

M. K. CHARYULU, University of Tulsa, and
J. B. SHEELER, Iowa State University

A theoretical mathematical model of stress distribution in a multi-layered system under loads of axial symmetry is presented. The model is generalized for any arbitrary number of horizontal layers and each layer is assumed to be linearly elastic, homogeneous, isotropic and of infinite extent in the horizontal plane, the last layer being semi-infinite. Vertical stress, vertical displacement and shear stress are considered to be continuous across any interface. In addition, the shear stress at an interface is assumed to be proportional to the relative displacement. Five sets of curves, obtained by an IBM 7074 computer, of vertical stress and displacement as a function of relative depth under parabolic loads are presented. A comparison of vertical stresses is made to those calculated by the theories of Burmister and Boussinesq.

•MATHEMATICAL models of vertical stress distribution for multi-layered systems have been presented by a number of authors. These models have assumed either that friction does not exist between the layers or that perfect adhesion exists between layers. This paper presents a model designed to represent a roadbed in which friction and relative displacement between layers are considered.

NOMENCLATURE

- a = radius of tire imprint;
- E = modulus of elasticity;
- H = distance from the upper surface of the system to an interface divided by a;
- j = subscript referring to the j^{th} layer;
- P = tire pressure;
- r, θ, z = cylindrical coordinates;
- u, o, w = displacements of a point in the r, θ, directions;
- β = proportionality constant between shear stress and relative displacement at an interface;
- ϵ = strain component;
- λ, μ = Lame's constants;
- σ = normal stresses;
- τ = shear stresses;
- X = stress functions; and
- ∇^2 = Laplace operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$.

THE MODEL

The model is generalized for any arbitrary number of horizontal layers and each layer is assumed to be homogeneous, isotropic, linearly elastic, and of infinite extent in the horizontal plane. The geometry and the physical properties may vary from

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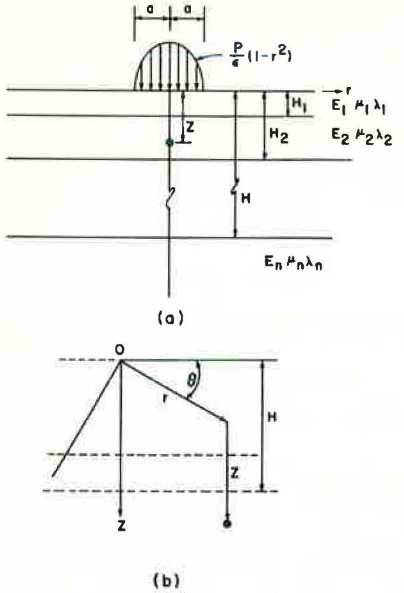


Figure 1.

layer to layer and the underlying layer is considered to be vertically semi-infinite. Interfacial conditions considered are continuity of vertical stress, vertical displacement and shear stress across the interface. The shear stress is assumed to be proportional to the relative displacement at the interface as has been experimentally verified for silty and sandy soils by Terzaghi and Peck (5). The unit weight of the material is assumed to be zero, consequently only stresses due to surface loads are found. Total stresses may be obtained by adding the stresses due to the column of material above the point in question.

The system is assumed to be subjected to a parabolically distributed vertical load in simulation of the contact pressure of truck tires. Lawton's research (3) has indicated that such a pressure distribution results from truck tires. Overloads tend to produce a pattern approaching uniform distribution.

The system is shown in Figure 1, and cylindrical coordinates are used to facilitate the solution of the problem since axial symmetry exists.

The general method of analysis involves the determination of a stress function for each layer. The stresses and displacements for the various layers are expressed in terms of the stress function which satisfied the boundary conditions of the layer in question. The problem is ultimately resolved (2) in a solution of the biharmonic equation:

$$\nabla^4 X = 0 \tag{1}$$

The following equations express the stresses and displacements in terms of the function X:

$$\sigma_r = \frac{\partial}{\partial r}\left[\lambda \nabla^2 X - 2(\lambda + \mu)\frac{\partial^2 X}{\partial r^2}\right] \tag{2}$$

$$\sigma_\theta = \frac{\partial}{\partial r}\left[\lambda \nabla^2 X - \frac{2}{r}(\lambda + \mu)\frac{\partial^2 X}{\partial r^2}\right] \tag{3}$$

$$\sigma_z = \frac{\partial}{\partial r}\left[(3\lambda + 4\mu)^2 X - 2(\lambda + \mu)\frac{\partial^2 X}{\partial z^2}\right] \tag{4}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(\lambda + 2\mu)^2 X - 2(\lambda + \mu)\frac{\partial^2 X}{\partial z^2}\right] \tag{5}$$

$$u = -\frac{(\lambda + \mu)}{\mu}\frac{\partial^2 X}{\partial r\,\partial z} \tag{6}$$

$$w = \frac{(\lambda + 2\mu)}{\mu} \nabla^2 X - \frac{(\lambda + \mu)}{\mu} \frac{\partial^2 X}{\partial r \, \partial z^2} \tag{7}$$

The boundary conditions based on the aforementioned assumptions are as follows:

At $z = 0$ $$\tau_{rz} = 0 \tag{8}$$

At $z = 0$ $$\sigma_z = p \left[1 - (r/a)^2 \right] \quad \text{for } o < r < a$$

$$\sigma_z = o \qquad \text{for } a < r < \infty \tag{9}$$

At any interface

$$\sigma_{zj} = \sigma z \, (j + 1) \tag{10}$$

At any interface

$$w_j = w_{j+1} \tag{11}$$

At any interface

$$\tau_{rz_j} = \tau rz \, (j + 1) \tag{12}$$

At any interface

$$\tau_{rz_j} = \beta \, (u_j - u_{j+1}) \tag{13}$$

As

$$z \longrightarrow \infty \qquad X \longrightarrow 0 \tag{14}$$

The determination of stresses and displacements lies in a solution (of Eq. 1) that satisfied the boundary conditions described by Eqs. 8 through 14. This was accomplished through Henkel transforms. The transformed boundary conditions produce sufficient equations to solve for the (4 n - 2) constants resulting from the solution to a system of n layers. The (4 n - 2) equations were written in matrix notation and numerical evaluation was accomplished by an IBM 7074 computer.

The program was developed for a parabolic distribution of load over a circular area and a four-layer system was used. Poisson's ratio was assumed to be 1/3 since an investigation by Peattie (4) found the probable value of Poisson's ratio to lie between 0. 3 and 0. 4 for granular road materials and bituminous materials.

Vertical stresses, σ_Z, and vertical displacements, w, were evaluated at the points of symmetry since maximum values exist on this line. The shear stress and the radial displacement on the line of symmetry are zero. The nondimensional parameters that were specified in the program are the thickness ratios and the elastic moduli ratios of all layers to the elastic modulus of the first layer.

TABLE 1

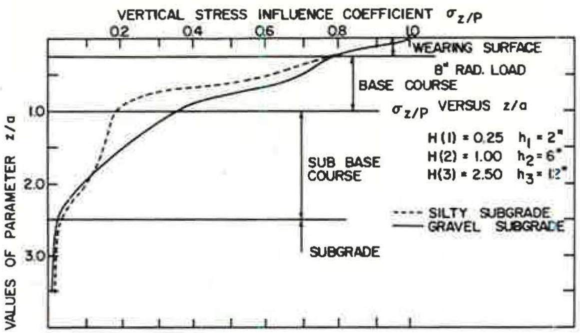
THICKNESS OF COMPONENTS OF PAVEMENT STRUCTURE

Tire imprint radius, in.	8	9	9	12	12
Surface course, in.	2	3	3	4	6
Base course, in.	6	6	9	8	12
Subbase, in.	12	18	12	16	18

TABLE 2

ASSUMED ELASTIC MODULI

Layer	Silt Subgrade (psi)	Gravel Subgrade (psi)
Surface course	5,000,000	5,000,000
Base course	1,000,000	500,000
Subbase	100,000	100,000
Subgrade	10,000	500,000



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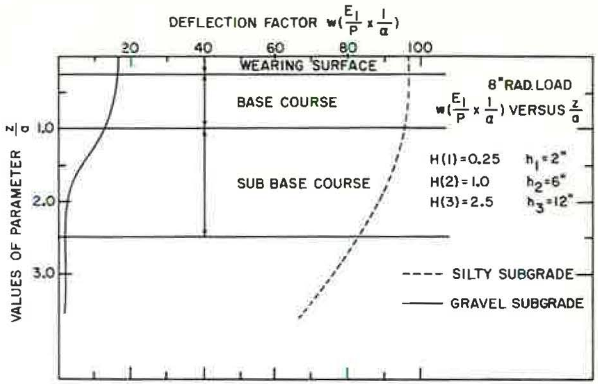

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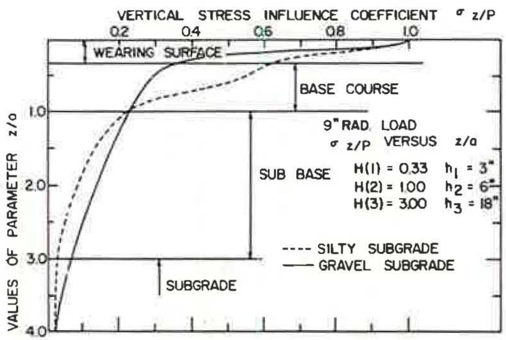

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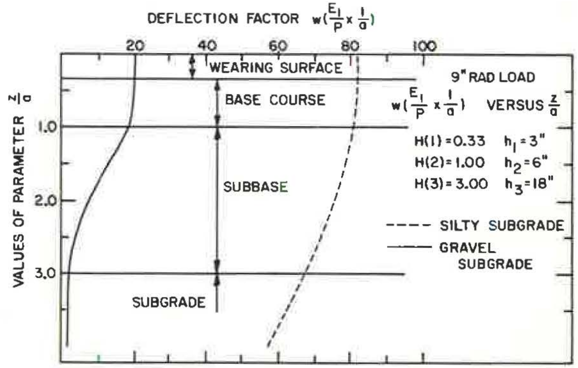

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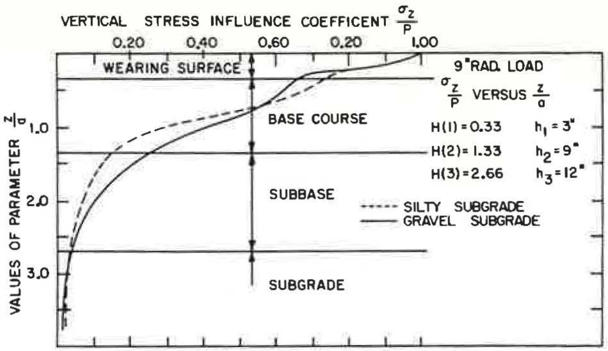

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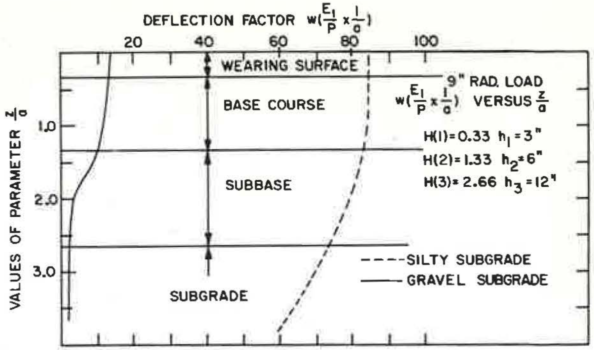

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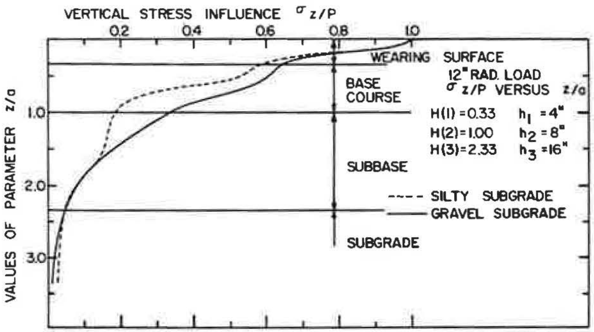

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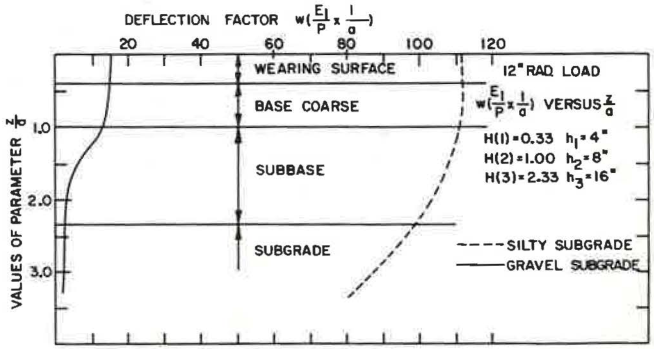

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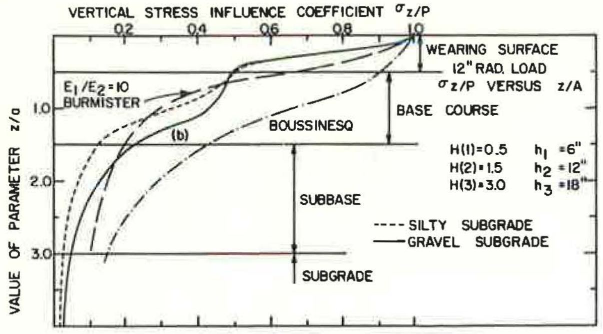

Figure 10.

Vertical stresses and displacements were evaluated for the thicknesses of pavement structural components over a silt subgrade and over a gravel subgrade (Tables 1 and 2).

The values of β, the proportionality constants between shear and displacement, were assumed to be tan 75° for the first interface and tan 85° for the second and third interfaces. The lower value was used for the first interface since it is generally believed that less binding exists at this interface than at deeper interfaces.

The results are plotted in Figures 2 through 11. Stresses at depth z are found by entering the graph at z/a (where a is the tire imprint radius) and determining the corresponding stress influence coefficient; the coefficient is then multiplied by P. Deflections are found in a similar manner from the deflection charts.

Details of the derivation of equations, Henkel transformation and matrix solution of this problem may be found elsewhere (2).

A comparison of stresses as found by the theories of Boussinesq and Burmister (1) with those resulting from this study is shown in Figure 10. The values at similar depths determined by this study and those of Burmister agree quite well, whereas those found by the Boussinesq theory are much higher.

Flexible pavements are generally constructed so that successively deeper layers have smaller moduli of elasticity. This type of construction causes the stresses and deflection at any depth to be reduced from those obtained in an ideal homogeneous bed. The results of this study indicate that higher stresses are produced in the base course than would be predicted by the Burmister theory. These findings tend to strengthen the concept that the base course is the main structural member of the pavement. There is also a suggestion that the interfacial shear strength has some bearing on the resultant vertical stresses, particularly at the first interface.

Much further study is needed, especially a comparison of actual stresses and deflections with those predicted by theory. Values of shear constants at the interfaces are guesswork at best, the assumed moduli of elasticity should be known with more accuracy and Poisson's ratio for soil materials should be more clearly understood before theoretical predictions can be made with much confidence. Further clarification of these factors and further study of the effect that the variables have on stresses and deflections should lead to more efficient and economic design criteria for flexible pavements.

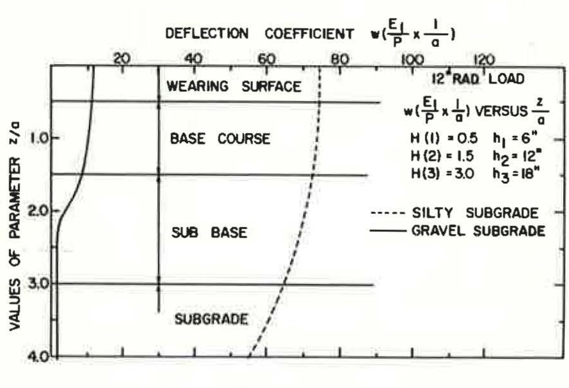

Figure 11.

ACKNOWLEDGMENT

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