# Retaining Wall Design—An Example of Small-Scale Optimization

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The feasibility of the application of optimization techniques to relatively small civil engineering problems as typified by retaining wall design is presented. The objective function as measured by the cost of concrete and reinforcing steel is minimized with respect to the toe, heel, stem base and footing thickness dimensions.

Four different numerical search techniques are employed in conjunction with a digital computer. They include: an exhaustive search, the converging gradient ascent, the steepest ascent and the random search methods. The relative efficiencies of the various approaches are discussed and compared.

Based on computer running time and programming effort required, it would appear that the state of the art has reached the point where very little additional expenditure is required in order to optimize the design of certain types of relatively small, common civil engineering problems.

•THE advent of the space age has injected a number of particularly meaningful words into the engineering vocabulary. In particular, the words system and optimum are two of the most commonly encountered. Confronted with vast projects composed of many interrelated components (system) involving huge expenditures of capital, the designer found it necessary and justifiable to consider the most efficient (optimum) configuration. The accumulated knowledge from this trend in conjunction with the general availability of digital computers has made it possible to examine the feasibility of applying optimal design to smaller systems.

If a physical system exhibits a structure which can be represented by a mathematical model, and if the value or merit of the system can be quantified as a function of the design variables, then some algorithm may be evolved for choosing the "best" answer. This procedure is generally referred to as mathematical programming.

The general programming problem seeks to minimize or maximize an objective function for n variables:

$$z = f(x_j);\ j = 1, \ldots n$$

subject to m inequality or equality constraints,

$$\begin{aligned} g_i(x_j) \geq = \leq b_i;\ i &= 1, \ldots m \\ j &= 1, \ldots n \end{aligned}$$

with feasible solutions

$$x_j \geq 0 \qquad j = 1, \ldots n$$

In other words, the vector of independent variable values which yields the largest or smallest value of the objective function is sought within the region bounded by constraints.

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As in most areas of analysis, the first successful approach to the general programming problem involved the linear case which can be stated as follows: $f(x_j)$ is a linear combination of the variables $x_j$ and each $g_i(x_j)$ is also a linear transformation. The algorithm, referred to as the simplex method, for solving the linear programming problem was first introduced by G. B. Dantzig (1) in 1948.

The second important class of programming problems falls under the heading of nonlinear programming. In this case $f(x_j)$ or at least one $g_i(x_j)$ is nonlinear in at least one $x_j$. The nonlinearity, as might be suspected, considerably increases the difficulty of obtaining a solution. A number of relatively formal methods have been proposed to solve the nonlinear case (2, 3, 4).

Often it may be necessary to resort to a numerical approximation when confronted with a nonlinear programming problem. Conceptually, the numerical approach is usually easier, yet computationally more demanding.

In order to examine the present feasibility of relatively small design optimization, three different numerical approaches were applied to a typical nonlinear civil engineering design problem—retaining wall design. The normal design of retaining walls using a digital computer has been previously discussed by Wadsworth (5).

## PROBLEM DEFINITION

The type of retaining wall chosen for analysis is shown in Figure 1. It is basically a cantilever wall with no key. The height of the wall (H) above subgrade, including the required depth (D) to avoid frost action, and the thickness of the stem at the top (TT) are given dimensions, along with the slope of the soil behind the wall and the necessary soil properties. The remaining four dimensions are taken as the design variables:

TB = thickness of stem at bottom,
TOE = toe length,
HEEL = heel length, and
B = base thickness.

There are many constraints which a cantilever retaining wall must satisfy, the first one being stability. Both overturning and sliding stability must be considered, with a minimum allowable factor of safety specified. The stability constraints used are those presented by Peck, Hanson and Thornburn (6) for a 1-ft length of wall. The remaining constraints are all of the requirements dictated by the American Concrete Institute Building Code Requirements for Reinforced Concrete - ACI 318-637.

The objective function measures the volume of concrete and the weight of reinforcing steel per lineal foot of wall and multiplies these values by their respective unit costs. The unit cost of ready-mix concrete and reinforcing steel are those for St. Louis, Missouri, obtained from a recent issue of Engineering News-Record (8).

Examination of the constraints and the objective function indicates that the retaining wall problem is nonlinear. For example, the cross sectional area of the footing involves products of the design variable thus resulting in a nonlinear combination. In addition, the objective function is not explicitly

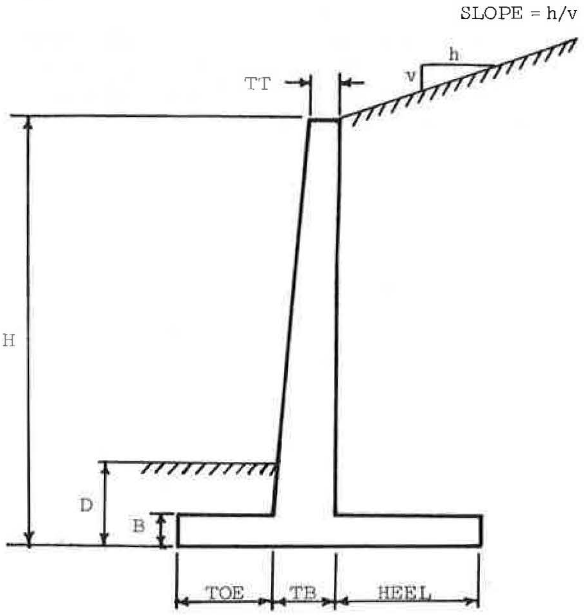

Figure 1. Cantilever retaining wall, no key.

TABLE 1

INPUT DATA

| Definition | Symbol | Value |
|---|---|---|
| Stem thickness at top, in. | TT | 12.000 |
| Total height, ft | H | 16.500 |
| Slope of soil surface behind wall | Slope | 3.000 |
| Soil depth in front of wall, ft | D | 3.500 |
| K-sub-V, ksf, $(k_v)$ | VK | 0.009 |
| K-sub-H, ksf, $(k_h)$ | HK | 0.033 |
| Unit weight of soil, kcf | GS | 0.125 |
| Unit weight of concrete, kcf | GC | 0.150 |
| Allowable soil pressure, ksf | QA | 3.000 |
| Friction between soil and base, ksf | SF | 0.550 |
| Overturning factor of safety | OFS | 1.500 |
| Sliding factor of safety | SFS | 1.500 |
| Compressive strength of concrete, psi | FC | 3750.0 |
| Allowable steel tensile stress, ksi | FS | 24.000 |
| Allowable concrete shear stress, psi | FSH | 67.000 |
| Modular ratio N, E(steel)/E(concrete) | EN | 8.250 |
| Balanced design J | FJ | 0.8781 |
| Balanced design R, psi | ARE | 271.00 |
| Unit price of re-bars, $/100 lb | US | 9.850 |
| Unit price of ready-mix, $/cu yd | UC | 13.500 |

| Bar Size | Bar Diameter (in.) | Bar Area (sq in) | Bar Perimeter (in.) | Bar Weight (lb/ft) |
|---|---|---|---|---|
| 3 | 0.375 | 0.11 | 1.178 | 0.376 |
| 4 | 0.500 | 0.20 | 1.571 | 0.668 |
| 5 | 0.625 | 0.31 | 1.963 | 1.043 |
| 6 | 0.750 | 0.44 | 2.356 | 1.502 |
| 7 | 0.875 | 0.60 | 2.749 | 2.044 |
| 8 | 1.000 | 0.79 | 3.142 | 2.670 |
| 9 | 1.128 | 1.00 | 3.544 | 3.400 |
| 10 | 1.270 | 1.27 | 3.990 | 4.303 |
| 11 | 1.410 | 1.56 | 4.430 | 5.313 |

expressed in terms of the design variables. These conditions strongly suggest the use of numerical optimization techniques.

The input data, design variables, constraints and objective function information are summarized in Tables 1 through 3. The actual wall to be optimized was again chosen from Peck, Hanson and Thornburn thus providing the input data. The concrete and steel specifications in the problem were replaced by the most recent reinforced concrete code and the reinforcing steel properties were those of standard deformed bars (ASMT A15, A305).

## METHODOLOGY

Essentially, three different search techniques were pursued for the purpose of comparing their relative efficiency for this type of application. In addition, an exhaustive search which examined all possible outcomes was conducted as a reference base.

A brief introduction to the three search methods is presented for explanatory purposes; however, the interested reader should consult a reference such as Wilde (9) to become aware of the pitfalls and limitations of each of the approaches.

The response surface is a plot of the objective function versus the independent variables. For the retaining wall problem the response surface would be a 5-dimensional plot of cost versus the 4 variable distances. The criterion for the existence of a maximum or minimum in an n + 1 dimensional space is that the gradient

TABLE 2

VARIABLES: RANGES AND INCREMENTS

| Variable | Integer Associated With Variable | Increment Size for Variable | Range of Variable | Corresponding Range of Associated Integer | Number of Increments |
|---|---|---|---|---|---|
| TB | ITB | 1.00" | 13"-30" | 1-18 | 18 |
| TOE | ITOE | 0.25' | 0.25'-8.00' | 1-32 | 32 |
| HEEL | IHEEL | 0.50' | 0.50'-16.00' | 1-32 | 32 |
| B | IB | 0.25' | 0.25'-3.00' | 1-12 | 12 |

Note: The number of possible combinations of the variables: 18 x 32 x 32 x 12 = 221, 184

In general: TB (in.) = ITB x 1.00 + TT
TOE (ft) = ITOE x 0.25
HEEL (ft) = IHEEL x 0.50
B (ft) = IB x 0.25

### TABLE 3
### CONSTRAINTS AND OBJECTIVE FUNCTION

| | |
|---|---|
| **Stability Constraints** | |
| Eccentricity of resultant soil reaction between center of base and toe, and also within kern | $0 \leq E \leq BASE/6$ |
| Soil pressure at toe non-negative and less than or equal to allowable | $0 \leq QTOE \leq QA$ |
| Soil pressure at heel non-negative and less than or equal to allowable | $0 \leq QHEEL \leq QA$ |
| Overturning factor of safety not less than minimum allowable | $AOFS \geq OFS$ |
| Sliding factor of safety not less than minimum allowable | $SFSMIN \geq SFS$ |
| **Concrete and Re-Bar Design Constraints** | |
| Effective depth not less than that required for shear | $DA \geq DS$ |
| Effective depth not less than that required for moment | $DA \geq DM$ |
| Clear space between re-bars not less than one inch | $CSPACE \geq 1$ |
| Clear space between re-bars not less than bar diameter | $CSPACE \geq BARD$ |
| Center-to-center spacing of re-bars not greater than three times the total concrete thickness | $SPACE \leq T3$ |
| Center-to-center spacing of re-bars not greater than 18 in. | $SPACE \leq 18$ |
| Actual bond stress not greater than allowable | $BOND \leq ABOND$ |
| **Objective Function** | |
| Minimize combined total cost of concrete and steel per lineal foot of cantilever retaining wall:<br>COST = Total volume of concrete per foot (cu yd) times unit cost of concrete ($/cu yd); plus total weight of steel per foot (100 lb), times unit cost of steel ($/100 lb). | |

$$\nabla f(\underline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = 0$$

Since the gradient is a vector quantity, it has both magnitude and direction. Its magnitude corresponds to the value of the "slope" of the response surface and its direction to the direction of the "steepest slope." Therefore, knowing the gradient at a point indicates the most efficient manner to proceed toward a "peak."

The gradient concept is at the heart of the "steepest ascent" technique. An initial point is chosen, the gradient calculated and a "step" is taken in the direction of the gradient. The process is repeated until a zero gradient is obtained which locates the optimum value of the response surface, if there is only one peak present in the region of interest.

For the retaining wall problem the gradient cannot be determined by partial differentiation because of the implicit nature of the objective function. Thus experiments were performed by evaluating the objective function in the neighborhood to determine the direction of steepest descent. For the 5-dimensional space, $5^4 = 625$, local experiments were performed at each intermediate point. Any point that was encountered which violated a constraint was simply assigned a large positive number to eliminate it from further consideration.

The converging gradient ascent technique embodies a similar philosophy except that each independent variable is searched sequentially while the remaining independent variables are fixed at their previously evaluated minimums.

The random search procedure as proposed by Brooks (10) has two attractive features which are pointed out by Wilde (9). First, no assumption about the form of the response surface need be made. Second, the probability p(f) of finding at least one solution in the best fraction f of the experimental region does not depend on the number of dimensions, for after n trials have been made at random,

$$p(f) = 1 - (1 - f)^n$$

or

$$n = \frac{\log [\,1 - p(f)\,]}{\log (1 - f)}$$

TABLE 4

NUMBER OF TRIALS n REQUIRED IN AN OPTIMUM-SEEKING EXPERIMENT CONDUCTED BY THE RANDOM METHOD, IN ORDER TO BE IN THE BEST FRACTION f WITH PROBABILITY p(f)

| f | p(f) | | | | |
|---|---|---|---|---|---|
| | 0.80 | 0.90 | 0.95 | 0.99 | 0.999 |
| 0.10 | 16 | 22 | 29 | 44 | 66 |
| 0.05 | 32 | 45 | 59 | 90 | 135 |
| 0.025 | 64 | 91 | 119 | 182 | 273 |
| 0.01 | 161 | 230 | 299 | 459 | 688 |
| 0.005 | 322 | 460 | 598 | 919 | 1379 |
| 0.001 | 1609 | 2302 | 2995 | 4603 | 6905 |

Reference (9).

The effectiveness of this relation is portrayed in Table 4. To apply the random search method, a value for a variable within its allowable range is chosen randomly with a "random number generator" subroutine. A different random number is selected for each independent variable in this manner and then the objective function is evaluated for the trial combination. After any number of trial solutions, the best combination of the independent variables found up to that point is known along with the current optimum value of the objective function. Thus after n trials, the current optimum has a probability p(f) of being within the best fraction f of all possible outcomes.

It should be noted that the simplest form of random search has been employed in this study. Significant efficiencies may be required as the dimensions of the problem increase. Socalled imbedding procedures (11) and the use of concepts from the area of statistical design of experiments (12) can be a worthwhile aid in this quest for efficiency.

## RESULTS

The results of the exhaustive search for the wall presented in Table 1 are summarized in Table 5. For the range of design variables chosen, only 12 percent of the possible walls did not violate the constraints. Examination of the ranges indicates an overextension thus including obviously inappropriate cases. However, the other extreme of too narrow a range based on intuition may in fact yield a sub-optimum answer.

TABLE 5

RESULTS OF EXHAUSTIVE SEARCH

| For Data of Table 1, 1963 Specifications | |
|---|---|
| The optimum cost per lineal foot of wall is | $17.28 |
| Determined in 221184 trials | |
| with 26048 walls satisfactory | |
| Optimum dimensions are TB = 1.1667 ft | B = 1.000 ft |
| TOE = 3.2500 ft | HEEL = 4.000 ft |
| Eccentricity E of resultant soil reaction is | 0.74 ft |
| Soil pressure at TOE QTOE is | 2.60≤3.00 ksf ...OK |
| Soil pressure at HEEL QHEEL is | 0.81≤3.00 ksf ...OK |
| Overturning factor of safety is | 2.59≥1.50.......OK |
| Sliding factor of safety is | 1.50≥1.50.......OK |
| Stem results | |
| Number 5 bars, at 3.50 in. center to center | |
| Cut off half of bars 5.32 ft from bottom of stem | |
| Cut off ¼ of bars 8.50 ft from bottom of stem | |
| Toe results | |
| Number 7 bars, at 10.00 in. center to center | |
| Heel results | |
| Number 8 bars, at 15.00 in. center to center | |

TABLE 6

| Search Technique | Converging Gradient Ascent | Steepest Ascent |
|---|---|---|
| Number of trials | 224 | 1875 |
| Number of successful trials | 138 | 883 |
| Computer run time | 45 sec | 2 min 10 sec |
| Minimum cost | $17.28 | $17.28 |

Peck, Hanson and Thornburn state that for best results, trial dimensions should be within the following proportions:

BASE/H = 0.40 to 0.65
TOE/BASE = approx. $^1/_3$
B/H = $^1/_{12}$ to $^1/_8$
Stem taper = $^1/_4$ to $^3/_4$ in./ft

The optimum wall ratios are:

BASE/H = 0.51
TOE/BASE = 0.39
B/H = approx. $^1/_{16}$
Stem taper = approx. $^1/_8$ in./ft

These results seem to indicate that the trial ratios might be revised using current specifications. However, caution should be exercised in view of the dependence of the result

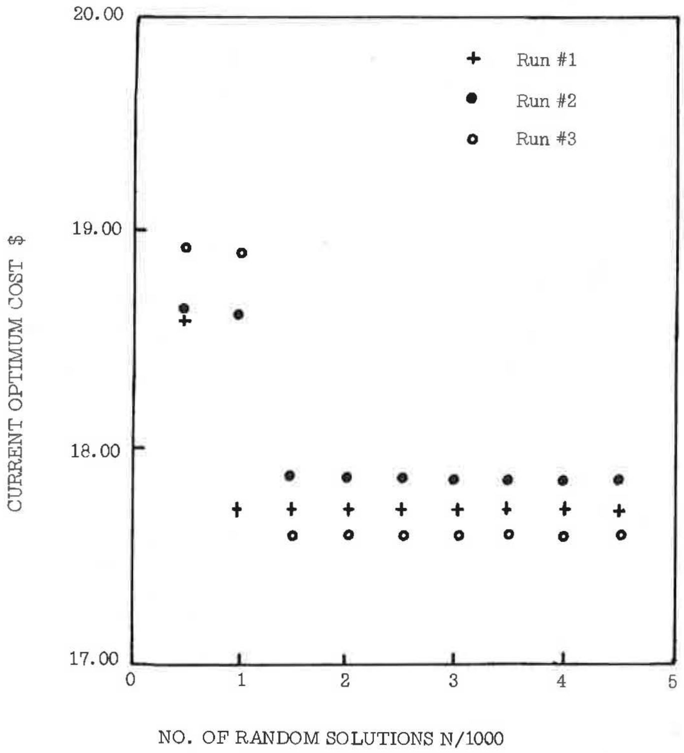

Figure 2. Random search results.

on the unit costs of the materials. The computer run time on a UNIVAC 1107 at the University of Notre Dame where the authors conducted the study for the exhaustive search was 25 min and 17 sec.

The results of the converging gradient ascent and steepest ascent search are compared in Table 6.

It should be remembered when examining these results that the relative efficiency is quite dependent on the proximity of the starting point to the optimum. In addition, the rapid convergence of the geometric techniques is quite dependent on the nature of the objective function being examined.

The results of three successive runs of 5000 trials each, using the random search approach are shown in Figure 2. All three runs yielded a solution with a cost within 5 percent of the optimum cost after 2000 trials. It should be noted that proximity to the optimum based on cost is not synonymous with the best fraction f in Table 4 because of the nonlinear distribution of number of walls in a particular price bracket. Based on a computation rate of approximately 800 trials per minute, the 2000 trial search was accomplished in approximately 2½ minutes. It is true that the unique optimum was not obtained in this case; however, the ease of application and flexibility tend to balance the limitations.

## CONCLUSIONS

In any optimization problem, the designer must balance the following factors:

1. Anticipated savings in construction costs resulting from an optimization over a conventional design;
2. Cost of computer running time; and
3. Cost of formulating the optimizing algorithm which is of course related to its sophistication.

With regard to the retaining wall example, it has been shown that all three optimization techniques involved computer running times of $^1/_{20}$ hr or less. At existing computer rates this becomes almost a trivial amount.

The cost of formulating and perfecting the computer program is less easily quantified. It is a function of the complexity of the technique and the talent of the programmer. The search techniques employed in this investigation are quite general and yet do not require any undue mathematical prowess. Also, the burden of program formulation will probably be considerably eased in the near future as general purpose optimization programs become available.

The economy of construction costs must of course be judged on an individual basis; however, in view of the previous discussion the benefits of optimization would not have to be large to justify the effort.

At the risk of possibly stating the obvious, there would appear to be a class of relatively small, common civil engineering design problems that can be optimized on an economically justifiable basis in a relatively straightforward manner in conjunction with digital computation.

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## *Discussion*

G. G. GOBLE, Associate Professor, Case Western Reserve University—This paper gives an interesting application of mathematical programming in civil engineering design. Since a substantial amount of work has been done in applying these methods in structural design, a listing of a few references may be useful to the reader. References (13) through (16) discuss the practical application of optimization in structural design. In references, (17)and (18) the use of linear programming is discussed. Nonlinear programming has been applied in references (19), (20) and (21) while in reference (22), the structural design problem is converted into an unconstrained minimization problem. This very brief list is by no means complete, but provides a review of the development of the field. In some of the references highly developed search techniques were used.

The writer agrees enthusiastically with the conclusion that optimizing techniques are ready for routine application in civil engineering design. Perhaps, the greatest impact will come from the freeing of the designer from many tedious design computations so that his effort can be spent on more creative tasks.

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