

# The Analysis of a Homogeneous, Cross-Anisotropic Elastic Half Space Undergoing Deformations That Possess a Vertical Plane of Symmetry

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This paper contains solutions for displacement, strain, and stress components produced throughout a homogeneous, cross-anisotropic elastic half space by loadings of a particular class. This class includes only loadings that are distributed over a circular area, and that give rise to a deformation field that has a vertical plane of symmetry. The loading conditions considered are unidirectional horizontal shear stresses, a linear variation in vertical direct stress, and a linear variation in vertical displacement.

•THE presence of patterns of structural anisotropy in soils is likely to be widespread and the reasons for this have been discussed previously (6). With this in mind a homogeneous, cross-anisotropic, elastic half space has been chosen as a simple model of earth masses because it possesses the symmetries that often exist in the structures of such masses. (A cross-anisotropic elastic body has an " $\eta$ " fold axis of symmetry or an axis of symmetry of rotation passing through each of its points; hence, its properties in all directions perpendicular to that axis are identical but different to the properties parallel to the axis.) The number of elastic parameters involved with this material is five instead of two as for an isotropic material.

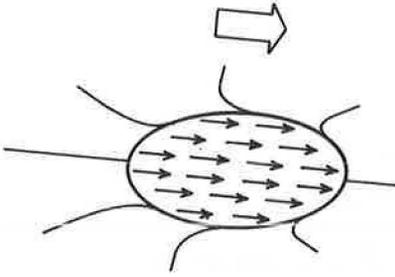
In this paper the class of deformations of a homogeneous, cross-anisotropic half space that have a vertical plane of symmetry are investigated. Three loading systems are considered (Fig. 1). The loads are applied over circular areas and are described briefly in the following.

•Loading Condition A—A uniform distribution of unidirectional horizontal shear stresses. Loading of this type produces a resultant horizontal thrust and is therefore typical of the braking and traction loads imposed on road pavements and horizontal loads developed in foundations due to the action of wind.

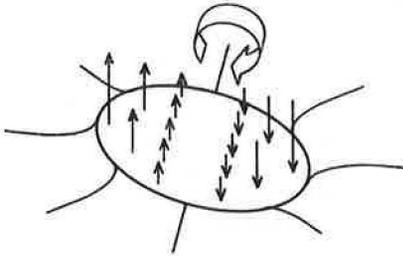
•Loading Condition B—A linear variation of vertical direct stresses. In this case the diameter along which the steepest gradient of vertical direct stress occurs is at right angles to the diameter along which the vertical direct stresses are zero. Since this type of loading develops a resultant moment but no resultant vertical force, it can therefore be used to analyze the action of moments on foundations.

•Loading Condition C—A linear variation of vertical displacement. This is analogous to loading condition B although the vertical displacement is defined rather than the vertical direct stress. Again there is a resultant moment but no resultant vertical force. Also the diameter along which the steepest gradient of vertical displacement occurs is at right angles to the diameter along which the vertical displacements are zero.

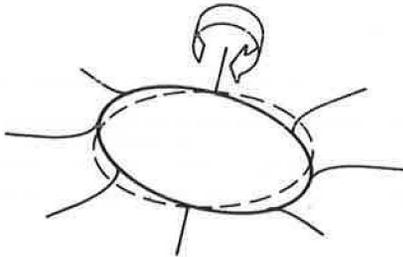
Half the loaded areas for both loading conditions B and C are required to withstand vertical tensile stresses. Such a requirement could be rarely met in practical soil mechanics and hence the main value of solutions for these conditions is when they are



A. Unidirectional horizontal shear stresses



B. Linear variation in vertical direct stress



C. Linear variation in vertical displacement

Figure 1. Loading conditions.

used in conjunction with solutions for a loading condition where there is a resultant vertical force of sufficient magnitude to prevent the development of vertical tensile stresses in the foundation. Two such loading conditions have been considered previously by Gerrard (6), who has given solutions for a total of five axisymmetric loadings applied to a homogeneous, cross-anisotropic elastic half space. These loadings are: (a) uniform vertical direct stress, (b) uniform vertical displacement, (c) linearly increasing radial shear stresses, (d) linearly increasing torsional shear stresses, and (e) linearly increasing torsional shear displacement.

Thus, it can be readily seen that a range of complex loading systems, applied over circular areas, can be analyzed by superimposing the solutions for these axisymmetric loading conditions onto the solutions for the three loading conditions considered in this paper.

The aim in providing these latter solutions has been to evaluate all displacement, strain and stress components at all points throughout the half space. The reason for this is that the most critically loaded zones from a soil mechanics point of view may not necessarily lie on the load axis. In addition it is often necessary to be able to estimate the total fields of any or all of the stresses and/or strains and/or displacements when assessing the behavior of loaded earth mass.

Loadings of a similar nature to those discussed in this paper have been previously considered by Barber (1, 2), Muki (8), and Westmann (13, 14). However, all of this work is restricted to isotropic materials and in addition the solutions presented are not as comprehensive as those given here.

However, the stresses produced in a cross-anisotropic half-plane by loadings in the form of either a vertically or horizontally acting line load have been calculated by De Urena

et al (4). In a further paper (9), these workers have integrated the above results to produce solutions for the cases of triangular distributions of vertical and horizontal loading stresses acting over a finite width. The respective resultant external loadings required in these cases would be (a) a combined moment and vertical thrust, and (b) a horizontal thrust. Thus these loading conditions are analogous to some of the loadings over circular areas considered in this paper and the previous work (6).

## NOTATION

### General

$r, \theta, z$	cylindrical coordinates
$u, v, w$	displacements in corresponding coordinate directions
$\widehat{rr}, \widehat{\theta\theta}, \widehat{zz}$	direct stress and shear stress components of the stress tensor
$\widehat{rz}, \widehat{\theta z}, \widehat{r\theta}$	

$\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}$	direct strain and shear strain components of the strain tensor
$\epsilon_{rz}, \epsilon_{\theta z}, \epsilon_{r\theta}$	
$a, b, c, d, f$	components of the elasticity tensor
$E_H$	modulus of elasticity in the horizontal direction
$E_V$	modulus of elasticity in the vertical direction
$\nu_H$	Poisson's ratio—effect of horizontal strain on complementary horizontal strain
$\nu_{HV}$	Poisson's ratio—effect of horizontal strain on vertical strain
$\nu_{VH}$	Poisson's ratio—effect of vertical strain on horizontal strain
$E$	modulus of elasticity—isotropic material
$\nu$	Poisson's ratio—isotropic material

### Quantities and Parameters Involved in Solutions

$$\alpha^2 = \frac{ad - c^2 - cf + f(ad)^{1/2}}{2fd}$$

$$\beta^2 = \frac{ad - c^2 - cf - f(ad)^{1/2}}{2fd}$$

$$\gamma^2 = \frac{a - b}{f}$$

$\delta$  = maximum vertical displacement (inches)

$$\sigma = \arcsin \left\{ \frac{2}{\left[ \psi^2 + (1 + r)^2 \right]^{1/2} + \left[ \psi^2 + (1 - r)^2 \right]^{1/2}} \right\}$$

$$\lambda = \arctan \left( \frac{2\psi}{\psi^2 + r^2 - 1} \right)$$

$$\phi = \alpha - \beta$$

$$\rho = \alpha + \beta$$

$$t_1 = \frac{d\alpha^2 - \frac{f}{2}}{\left(c + \frac{f}{2}\right)\alpha}$$

$$t_2 = \frac{d\alpha^2 + \frac{f}{2}}{\left(c + \frac{f}{2}\right)\alpha^2}$$

$$m_1 = \frac{\left(c + \frac{f}{2}\right)(\alpha^2 - \beta^2)}{f\beta} = \frac{\left(c + \frac{f}{2}\right)\rho\phi}{\frac{f}{2}(\rho - \phi)}$$

$$m_2 = \frac{\left(c + \frac{f}{2}\right)}{\frac{f}{2}} \cdot \frac{\alpha^2}{c + d\alpha^2}$$

$$m_3 = -\frac{2^{1/2}}{\pi^{1/2}} \cdot \frac{(c + d \rho^2)(c + d \phi^2)}{d(\rho^2 - \phi^2)}$$

$$m_4 = -\frac{2^{1/2}}{\pi^{1/2}} \cdot \frac{c + d \alpha^2}{2d\alpha}$$

$$\eta = \left[ (\psi^2 + r^2 - 1)^2 + 4\psi^2 \right]^{1/4}$$

$p$  = maximum vertical loading pressure (psi)

$$q_1 = \frac{d\phi^2 - \frac{f}{2}}{\left(c + \frac{f}{2}\right)\phi}$$

$$q_2 = \frac{d\rho^2 - \frac{f}{2}}{\left(c + \frac{f}{2}\right)\rho}$$

$r$  = horizontal offset from load axis/ $r_0$  (inch/inch)

$r_0$  = loaded radius (inches)

$s$  = unidirectional horizontal shear stress (uniform distribution) (psi)

$z$  = depth from surface/ $r_0$  (inch/inch)

#### STRESS-STRAIN RELATIONSHIPS

For a cross-anisotropic elastic material the stresses can be expressed in terms of the strains by:

$$\widehat{r\bar{r}} = a \epsilon_{rr} + b \epsilon_{\theta\theta} + c \epsilon_{zz} \quad (1a)$$

$$\widehat{\theta\bar{\theta}} = b \epsilon_{rr} + a \epsilon_{\theta\theta} + c \epsilon_{zz} \quad (1b)$$

$$\widehat{z\bar{z}} = c \epsilon_{rr} + c \epsilon_{\theta\theta} + d \epsilon_{zz} \quad (1c)$$

$$\widehat{r\bar{\theta}} = (a - b) \epsilon_{r\theta} \quad (1d)$$

$$\widehat{r\bar{z}} = f \epsilon_{rz} \quad (1e)$$

$$\widehat{\theta\bar{z}} = f \epsilon_{\theta z} \quad (1f)$$

The relations expressing the direct strains in terms of the direct stresses together with others expressing the elasticity components  $a$ ,  $b$ ,  $c$ ,  $d$  in terms of the Young's moduli and Poisson's ratios have been given previously (6).

The strain components are

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \quad (2a)$$

$$\epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad (2b)$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \quad (2c)$$

$$2 \epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad (2d)$$

$$2 \epsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r} \cdot \frac{\partial w}{\partial \theta} \quad (2e)$$

$$2 \epsilon_{r\theta} = \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (2f)$$

#### METHOD OF SOLUTION AND PRESENTATION OF RESULTS

The method of solution of problems involving the static loading of layers of cross-anisotropic elastic material, previously outlined by Gerrard and Mulholland (7), is based on the integral transform techniques developed by Sneddon (10) and Tranter (11). The problem considered in this paper, i. e., a homogeneous half space undergoing deformations that possess a vertical plane of symmetry, is a special case of the general range of problems that may be solved by this method.

For all three loading conditions the solutions for the displacements, strains, and stresses involve integrals of products of Bessel functions. In order to simplify presentation, these integrals have been abbreviated according to the following notation:

$$I(\eta, \tau, \chi, \psi) = \int_0^{\infty} J_{\eta}(k) \cdot J_{\tau}(kr) \cdot k^{\chi} \cdot e^{-\psi k} \cdot dk \quad (3a)$$

$$J(\eta, \tau \pm \mu, \chi, \psi) = \int_0^{\infty} J_{\eta}(k) \left[ \frac{J_{\tau}(kr) \pm J_{\mu}(kr)}{2} \right] \cdot k^{\chi} \cdot e^{-\psi k} \cdot dk \quad (3b)$$

The form of solution obtained for the types of deformation considered has been shown to depend on the parameters  $\alpha^2$  and  $\beta^2$  that are functions only of the elastic constants (7). Hence, for each of the loading conditions, three complete lists of solutions are given for the following cases:

1. Cross-anisotropic,  $\alpha^2$  positive,  $\beta^2$  positive;
2. Cross-anisotropic,  $\alpha^2$  positive,  $\beta^2$  zero;
3. Isotropic; this is a particular case of item (2) with  $\alpha^2 = 1$ ,  $\beta^2 = 0$ .

Appended to the solutions for each loading condition is a section dealing with the evaluation of the relevant integrals. These sections are divided into three parts depending upon whether the integrals are to be evaluated:

1. At a general point within the system (i.e.,  $r \neq 0$ ,  $\psi \neq 0$ );
2. At a point on the load axis ( $r = 0$ ); or
3. At a point on the surface ( $\psi = z = 0$ ).

With regard to the evaluation of the integrals at a general point it can be seen from the lists that simple and direct results are obtained for loading condition C where the surface displacements are defined. However, this is not so for the defined stress conditions (A and B) where the integrals had to be evaluated by numerical integration, on a high-speed digital computer, for a range of values of  $r$  and  $\psi$ .\* Some of these integrals have been previously tabulated by Eason, Noble, and Sneddon (5) for different ranges of values of  $r$  and  $\psi$  than were used in the current work. For anisotropic materials the parameter  $\psi$  is in the form of either  $\rho z$ ,  $\phi z$ ,  $\alpha z$ , or  $\gamma z$  and therefore is a function of both the elastic parameters and the depth. This means that when using the tables\* to calculate the stresses, strains, and displacements at a particular point within the system it will be generally necessary to interpolate between the tabulated points along the  $\psi$  coordinate.

\*The tabulated values of a total of 22 such integrals are in Tables 1 through 22, which are included in a lengthy Appendix not published in this Record, but which is available from the Highway Research Board at the cost of reproduction and handling. Refer to XS-24, Highway Research Record 282.

At points on the load axis (i. e.,  $r = 0$ ) the integrals involved in all of the loading conditions can be evaluated simply and directly. On the other hand for points on the surface (i. e.,  $\psi = z = 0$ ) the integrals for all loading conditions become discontinuous in nature, in order to fulfill the loading conditions. As indicated most of the integrals have simple results while the remainder yield results in the form of hypergeometric functions (10, 12). Integrals whose coefficients contain  $z'$  have not been considered since their products ( $z \cdot I$  or  $z \cdot J$ ) are in general zero for points on the surface. The values of integrals given in this paper are based directly or indirectly on the results obtained by Watson (12), Sneddon (10) and Bateman Manuscript Project (3).

In all solutions the orientation reference in any horizontal plane is given by  $\theta = 0$  or  $\pi$  along the direction of the vertical plane of symmetry in the displacement patterns.

#### TOTAL MOMENT-DEFINED DISPLACEMENT RELATIONSHIP

For loading condition C it is possible to derive relationships between the defined surface displacement and the requisite applied moment. These relationships, found by integrating the moment of the vertical direct stress over the loaded area, are as follows:

Cross-anisotropic:  $\alpha^2$  positive,  $\beta^2$  positive

$$\text{Total moment} = -\frac{8}{3} \cdot \delta \cdot r_0 \cdot \frac{(c + d\rho^2)(c + d\phi^2)}{d(\rho + \phi)\rho\phi} \cdot \frac{\frac{f}{2}}{c + \frac{f}{2}} \quad (4a)$$

Cross-anisotropic:  $\alpha^2$  positive,  $\beta^2$  zero

$$\text{Total moment} = -\frac{8}{3} \cdot \delta \cdot r_0 \cdot \frac{(c + d\alpha^2)^2}{2d\alpha^3} \cdot \frac{\frac{f}{2}}{c + \frac{f}{2}} \quad (4b)$$

Isotropic

$$\text{Total moment} = -\frac{8}{3} \cdot \delta \cdot r_0 \cdot \frac{E}{2(1 - \nu^2)} \quad (4c)$$

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#### REFERENCES

1. Barber, E. S. Shear Loads on Pavements. Public Roads, Vol. 32, No. 6, 1963.
2. Barber, E. S. Shear Loads on Pavements. Public Roads Vol. 33, No. 8, 1965.
3. Bateman Manuscript Project. Tables of Integral Transforms. McGraw-Hill, 1954.
4. DeUrena, R., Piquer, J. S., Muzas, F., and Sanz Saracho, J. M. Stress Distribution in Cross-Anisotropic Media. Proc. 1st Cong. Int. Soc. Rock Mech., Lisbon, Vol. 1, p. 313-317, 1966.
5. Eason, G., Noble, B., and Sneddon, I. N. On Certain Integrals of the Lipschitz-Hankel Type Involving Products of Bessel Functions. Phil. Trans., Series A, Royal Soc. London, Vol. 247, p. 529-551, 1955.
6. Gerrard, C. M. The Axisymmetric Deformation of a Homogeneous, Cross-Anisotropic Elastic Half Space. Highway Research Record 223, p. 36-44, 1968.
7. Gerrard, C. M., and Mulholland, P. Hexagonally Anisotropic Layered Elastic Systems. Submitted to ASCE (Eng. Mech. Div.), 1967.

8. Muki, R. Asymmetric Problems of the Theory of Elasticity for a Semi-Infinite Solid and a Thick Plate. Progress in Solid Mechanics, Vol. 1, North Holland Pub. Co., Amsterdam, 1960.
9. Piquer, J. S., Muzas, F., De Urena, R., and Grajera, F. Foundations in Cross-Anisotropic Ground. Proc. 1st Cong. Int. Soc. Rock Mech., Lisbon, Vol. 1, p. 531-536, 1966.
10. Sneddon, I. N. Fourier Transforms. McGraw-Hill, 1951.
11. Tranter, C. J. Integral Transforms in Mathematical Physics. Methvens Monographs on Physical Subjects, 3rd Ed., 1966.
12. Watson, G. N. The Theory of Bessel Functions. 2nd Ed., Cambridge Univ. Press, 1944.
13. Westmann, R. A. Layered Systems Subjected to Asymmetric Surface Shears. Proc. Roy. Soc., Edinburgh, Vol. LXVI, Series A, Part 3, p. 140, 1963.
14. Westmann, R. A. Layered Systems Subjected to Surface Shears. Proc., ASCE, Eng. Mech. Div., Vol. 89, No. 6, p. 177, 1963.