Time and Load Independent Properties of Bituminous Mixtures

S. A. SWAMI, West Virginia Institute of Technology; and
W. H. GOETZ and M. E. HARR, Purdue University

This paper presents a new approach to characterize the time-dependent behavior of asphaltic concrete by a transfer function that is derived from the frequency response of the material, using sinusoidal loading tests. The ratio of the Laplace transform of the output to the Laplace transform of the input to any system is defined as the transfer function of the system between the input and the output. The transfer function is a function of the complex variable $s = j\omega$, where ω is the frequency of the input and j is the imaginary unit. In this study asphaltic concrete was treated as a damped, linear system in which the applied force f(t) was the input and the resulting displacement x(t) was the output. Both the input and output were functions of time. By varying the frequency of the input, the output-input magnitude ratio was obtained for each frequency, and its plot against log frequency yielded the frequency spectrum. Also, the time lag between the output and the input was noted from which the phase angle was calculated. A simple geometric technique to obtain the transfer function directly from the frequency spectrum was used in this investigation. This technique was applied to the results of the test specimens of asphaltic concrete cut from laboratory compacted beams of 2 different compositions tested at 3 temperatures. It is shown that the transfer function for bituminous concrete thus derived from a dynamic test can be used to predict the displacement of the test specimen subjected to a static load by treating the latter as a step function of time and through the use of Laplace inverse transform. The excellent agreement between the calculated and measured values of the displacements indicates that the transfer function represents a material property that is independent of load input. The concept of transfer function makes it possible to represent the time-dependent behavior of asphaltic concrete by a fourth order linear differential equation with constant coefficients without assuming any spring-dashpot model. The coefficients can be computed from the roots of the denominator and numerator of the transfer function.

•THEORETICAL DEVELOPMENTS in the stress-displacement behavior of asphaltic concrete in a pavement structure are very restricted in their applicability to represent actual field conditions because of the complexity of asphaltic concrete as a structural material and the complicated interaction with the other structural layers. A rigorous theoretical solution to a typical boundary value problem in pavement mechanics must satisfy the equations of equilibrium (or motion), compatibility equations (or some other form of conservation equation), boundary conditions, and initial conditions. The development of such a solution requires the use of some form of stress-displacement-time relation for the material, and it is this aspect that specifies the solution for a particular

Paper sponsored by Committee on Mechanical Properties of Bituminous Paving Mixtures and presented at the 49th Annual Meeting.

medium. At present, there are no satisfactory constitutive relations available for pavement materials, both asphaltic concrete and soils, and this is perhaps the greatest impediment to realistic theoretical solutions of the response of pavement systems to applied loads, dynamic or static.

Having the ultimate objective to design such layered pavement structures, the theory of linear viscoelasticity has been widely used for the evaluation of pavement components as well as for understanding their response to varied loading conditions. Layered systems were not, however, analyzed using time-dependent (viscoelastic) material properties until after the development of the correspondence principle for isotropic media in 1955 by Lee (1) and shortly thereafter the extension by Biot (2) to include anisotropic media. Stress and strain for linear viscoelastic materials can be related by either differential or integral linear operators. The differential operator form of the stress-strain law is most commonly used and, as Biot (3) has pointed out, may be visualized as a combination of springs and dashpots. More general methods of viscoelastic stress analysis have been described by Lee (4) and Lee and Rogers (5).

Pister and Monismith (6) have applied the basic differential equation for the 3-element model to the solution of a viscoelastic beam on an elastic foundation. Pister (7) has considered the solution of a viscoelastic plate on a viscoelastic foundation. Other literature (8, 9, 10) also exists in limited quantity dealing with viscoelastic slabs or foundations in various ways. Harr (11) used a 2-element model to show the influence of vehicle speed on pavement deflection. The usefulness of these methods is restricted to the extent of the realistic representation of the viscoelastic characteristics of the materials of the pavement structure in the numerical analysis.

The study of viscoelastic characteristics of bituminous mixtures has been the object of many research workers, and here again spring-dashpot models have been generally made use of. By using a Burger's model, Wood and Goetz (12) found that the behavior of a sheet asphalt mixture obeyed the laws of linear viscoelasticity for limited stresses and small deformations. Secor and Monismith (13) observed that the 4-element model, suggested by Kuhn and Rigden (14), was capable of representing the displacement of an asphaltic concrete only to a limited extent.

Dynamic tests, such as those employing sinusoidal loading, have been used to study the viscoelastic response of asphaltic concrete by some investigators (15, 16, 17). They have attempted to characterize the viscoelastic response by using the phase angle and the complex elastic modulus, obtained from the input-output traces of the dynamic tests. However, these parameters are not easily adaptable for numerical analysis. Besides, they are functions of the input load, which seriously limits the ability of these parameters to truly represent the load response characteristics of the material.

This motivated the authors to search for a parameter or parameters or a function that can represent the viscoelastic response of the material under any load and that, at the same time, is independent of the type and magnitude of the input load. Such a function should necessarily be time-dependent and should be used as the material characteristic for both static and dynamic loads. A study of the literature reveals that transfer functions are frequently used to characterize a dynamic system in electrical and mechanical engineering problems (18, 19, 20, 21, 22, 23). While serving as a link between the time-dependent input and output of the system, the transfer function completely describes its dynamic response. Hence, it was attempted to investigate the feasibility of applying the concept of transfer functions to describe the response of bituminous mixtures to dynamic and static loads. This paper presents a method of obtaining the transfer function for a given bituminous concrete from laboratory tests and studies the extent to which this function can represent the viscoelastic characteristics of the material. Effects of mix type, temperature, and anisotropy on the transfer function were studied with a view to providing insight into this function. It is hoped that this insight may facilitate the development of more refined methods for the structural design of pavements.

It is assumed in this investigation that asphaltic concrete behaves as a linear system. Although it has been recognized that the behavior of this material is nonlinear, particularly at higher stress levels and at higher temperatures, the investigations of Busching, Goetz, and Harr (24) and others (15, 25) have established the validity of this assumption for small displacements normally encountered in bituminous mixtures.

CONCEPT OF TRANSFER FUNCTION

The ratio of an operational output (Laplace transform of the output) of a dynamical system to the operational input (Laplace transform of the input) is called the transfer function between the operational input and its corresponding output. The transfer function is shown schematically in Figure 1. It is by definition a function of the complex variable s.

Figure 1. The transfer function.

If a rigid body is subjected to a dynamical force f(t), then the force and the resulting displacement x(t) can be considered as the input and output respectively for the dynamic system. The transfer function between the operational force and the operational displacement is given by

$$G(s) = \frac{\bar{x}(s)}{\bar{f}(s)} \tag{1}$$

where $G(s)$ = the transfer function,
$\bar{x}(s)$ = Laplace transform of the output, $\mathcal{L}[x(t)]$, and
$\bar{f}(s)$ = Laplace transform of the input, $\mathcal{L}[f(t)]$.

Solving for $\bar{x}(s)$,

$$\bar{x}(s) = G(s)\bar{f}(s) \tag{2}$$

The inverse transform of $\bar{x}(s)$ is x(t) and is

$$x(t) = \mathcal{L}^{-1}\left[G(s)\bar{f}(s)\right] \tag{3}$$

Equation 3 shows that, once the transfer function G(s) is known for any system, the displacement x(t) can be evaluated for that system for another given input force f(t).

In general, the transfer function G(s) is a ratio of 2 polynomials in s, such as

$$G(s) = \frac{N(s)}{D(s)} = \frac{(s - a_1)(s - a_2)\ldots(s - a_n)}{(s - b_1)(s - b_2)\ldots(s - b_m)} \tag{4}$$

where a_i's and b_i's are the roots or zeros of the numerator N(s) and denominator D(s) of G(s) respectively. The b_i's are also called the poles of G(s) because the transfer function becomes infinite when evaluated at a zero of its denominator. That is,

$$G(b_i) = \infty \tag{5}$$

Substituting for G(s) in Eq. 2, $\bar{x}(s)$ can be rewritten as

$$\bar{x}(s) = \frac{\bar{f}(s)(s - a_1)(s - a_2)\ldots(s - a_n)}{(s - b_1)(s - b_2)\ldots(s - b_m)} \tag{6}$$

Equation 6 can be expanded in m partial fractions for the m roots of D(s) as

$$\bar{x}(s) = \frac{C_1}{(s - b_1)} + \frac{C_2}{(s - b_2)} + \ldots + \frac{C_m}{(s - b_m)} \tag{7}$$

which is an identity in s where C_1, C_2, ..., C_m are constants independent of s. Using the Laplace inverse transform, the final solution for x(t) is obtained from Eq. 7 as

$$x(t) = C_1 e^{b_1 t} + C_2 e^{b_2 t} + \ldots + C_m e^{b_m t} \tag{8}$$

From this discussion it is obvious that once the transfer function for a system is known, the displacement x(t) of the system for a given force input f(t) can be evaluated theoretically.

DETERMINATION OF THE TRANSFER FUNCTION

The experimental basis to determine the transfer function for a test specimen is the frequency spectrum, which, in turn, may be obtained from sinusoidal input-output data of the test specimen (23). The plot of the amplification (that is, the ratio of the output magnitude to the input magnitude) against the test frequency is the frequency spectrum, and for convenience the magnitude of the amplification is plotted in decibels ($R_{db} = 20 \log_{10} R$). The frequency spectrum is approximated with a series of connected straight lines so as to form asymptotes to the curve (Fig. 2). The intersections of the asymptotes determine the corner frequencies. The modified transfer function is then obtained (26) as a function of the slopes of the asymptotes and their corner frequencies as

$$G(j\omega) \cong A(j\omega + \omega_1)^{\frac{k_1}{6}} (j\omega + \omega_2)^{\frac{k_2-k_1}{6}} (j\omega + \omega_3)^{\frac{k_3-k_2}{6}} \ldots (j\omega + \omega_n)^{\frac{k_n-k_{n-1}}{6}} \tag{9}$$

The modified transfer function is the vector amplification or simply the ratio of the magnitude of the output to that of the input and is a function of the input frequency. By substituting $s = j\omega$ in Eq. 9, the transfer function for the system is obtained.

$$G(s) \cong A(s + \omega_1)^{\frac{k_1}{6}} (s + \omega_2)^{\frac{k_2-k_1}{6}} (s + \omega_3)^{\frac{k_3-k_2}{6}} \ldots (s + \omega_n)^{\frac{k_n-k_{n-1}}{6}} \tag{10}$$

The constant A can easily be determined from any one experimental frequency in the frequency spectrum by calculating the absolute value of the transfer function from Eq. 9 at that frequency and equating it to the corresponding amplification value in the frequency spectrum.

In approximating the frequency spectrum, the slopes of the asymptotes are generally taken to be 6 decibels per octave for greater ease of determination, although straight

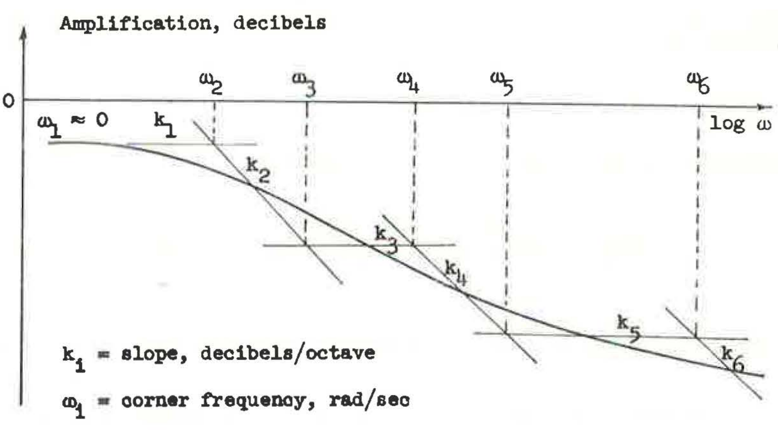

Figure 2. Asymptotic approximation of a frequency spectrum.

lines of any slope may be used. (An octave is defined as the frequency range for which the ratio of the upper bound frequency to the lower bound frequency is 2 to 1; that is, $\omega_i < \omega < 2\omega_i$ is an octave.) The closer the approximation is to the actual frequency spectrum curve, the better will be the approximation of the transfer function being sought.

EXPERIMENTAL INVESTIGATION

The experimental phase of this study had as an objective the development and use of accurate techniques for obtaining all of the data required in the determination of the transfer function from a frequency spectrum. The core of the technique lies in devising a system that can apply a sinusoidal force input of a desired magnitude and frequency and that can simultaneously measure the displacement output.

An MTS electronic function generator coupled with a loading frame that was fitted

Figure 3. Function generator.

Figure 4. Loading frame.

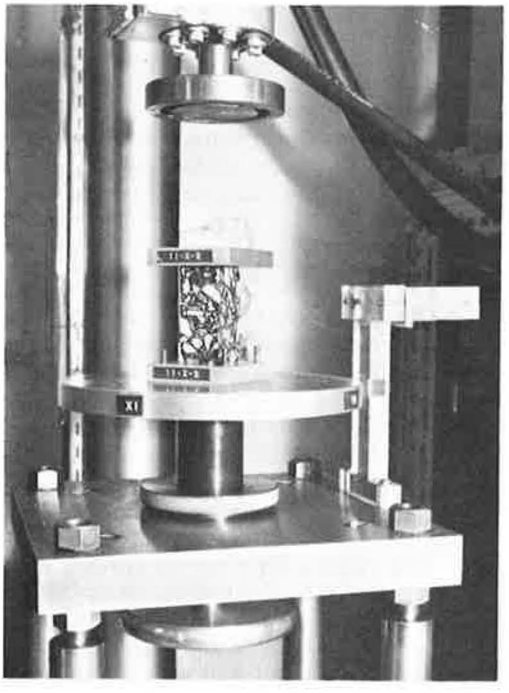

Figure 5. Specimen in position for testing.

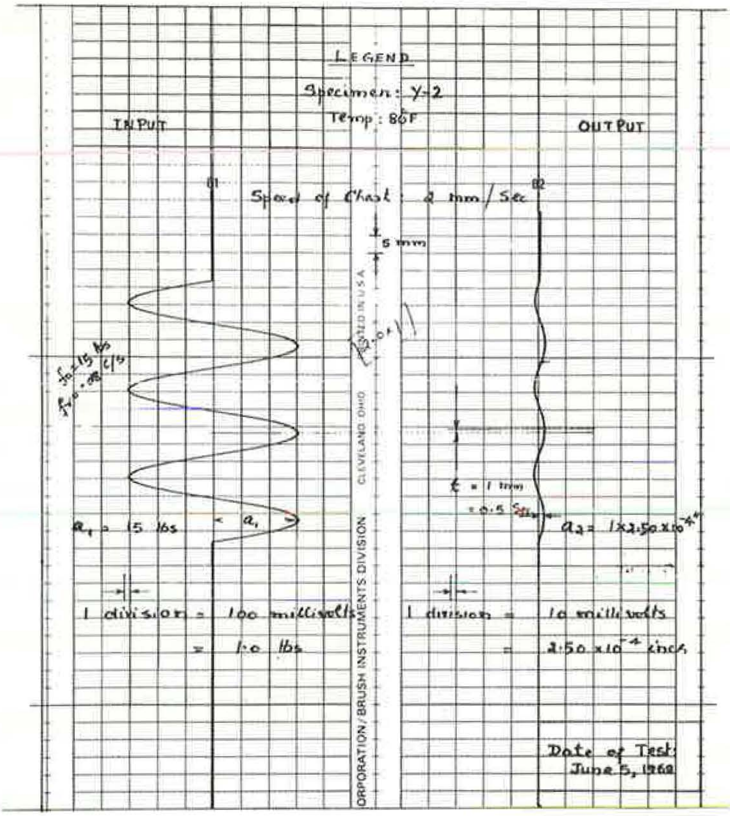

Figure 6. Typical traces of a sinusoidal loading test.

with an electronically controlled hydraulic actuator was used for this purpose (Figs. 3 and 4). The loading frame was installed in a constant temperature room that facilitated carrying out the tests at the desired temperatures. A 2-channel Brush recorder that was attached to the function generator recorded the input and output of the test specimen simultaneously.

Dynamic Tests

The function generator was used to apply sinusoidal force inputs to the asphaltic concrete specimen by suitable manipulation of the controls. The magnitudes of the force varied from 2.5 to 15 lb and the frequencies varied from 0.0016 to 1.6 cycles per second.

Because in a sine test the specimen is alternatively in compression and in tension, the method of mounting the specimen in the test assembly was of paramount importance. Two aluminum plates, 2 by 2 by $^3/_8$ in. in size, were glued to the specimen top and bottom respectively, using commercially available Eastman 910 adhesive. The plates, in turn, were connected to the top and bottom platens by screws and nuts. The top platen was rigidly fixed to the moving ram of the hydraulic actuator while the bottom platen was fixed to the loading deck (Fig. 5). Remotely controlled and commanded by the electronic console of the function generator, the hydraulic ram moves up and down at the preset sine force and frequency. Thus the specimen was subjected to the desired sinusoidal force input that was recorded by the left channel of the Brush recorder. The corresponding displacement of the specimen was sensed by the LVDT in the actuator

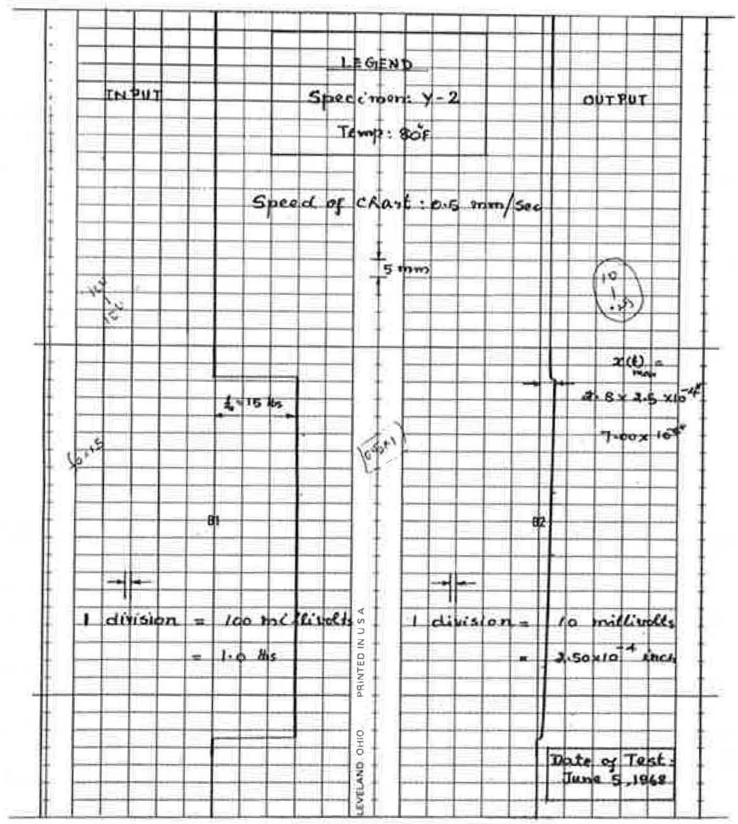

Figure 7. Typical traces of a static compression test.

and was simultaneously recorded as the output in the right channel of the recorder. Figure 6 shows the 2 traces thus obtained in a typical dynamic test.

Static Tests

The same test assembly that was used for the dynamic tests was used for the static tests. When the function generator is set for a combination of a ramp function and a high frequency, the actuator applies a static compressive force of desired magnitude. As before, the left and right channels recorded the input and output respectively. The 2 traces of a typical static compression test are shown in Figure 7.

Materials and Preparations of Specimens

Two different gradings, one with a maximum size of the No. 4 sieve and the other the $\frac{3}{8}$-in. sieve, were used in this investigation. The bituminous mixes prepared on the basis of these gradings were designated as mix 1 and mix 2 respectively. A 60 to 70 penetration asphalt cement was used. The mix was compacted into beams 2 by $2\frac{1}{2}$ by 12 in., from which test specimens 1 by 1 by 2 in. were cut along the x, y, and z axes of the beam, x being the longest axis and z the vertical axis. The specimens were designated as x-1, y-1, z-1, x-2, y-2 and z-2, the letter denoting the axis along which it was cut and the numeral the mix. The details of obtaining the specimens are given in another report (26).

DISCUSSION OF RESULTS

Frequency Response of Asphaltic Concrete

In most of the dynamic tests a magnitude of 10 lb was employed for the sinusoidal load input for frequencies covering 3 decades, varying from 0.01 radians per sec to 10.0 radians per sec (rad/sec). These were observed to be respectively the slowest and the fastest frequencies to which the test specimens were responsive. The input magnitude and the output displacement were recorded in pounds and inches respectively. Also, both of them were recorded in millivolts (mv).

The ratio of the output to the input expressed in decibels (also referred to as amplification or gain) plotted against log frequency gives the desired frequency spectrum. The phase lag or simply the phase angle for each frequency was also computed from

$$\phi = \omega t \tag{11}$$

where ϕ = the phase angle in radians,
ω = the test frequency in rad/sec, and
t = time lag between input and output in seconds.

The phase angle is also plotted against log frequency. The 2 plots together represent the frequency response of the test specimen completely.

Table 1 shows typical test results for a dynamic test, and Figure 8 gives the frequency response for the same test. It is seen that the displacement of the specimen is largest at the slowest frequency and is least at the fastest frequency. Thus the gain continuously falls with increase in frequency, and this indicates the overdamped nature of the test specimen.

The phase angle increases with increase in frequency in the first decade of test and then decreases so that a bell-shaped curve results when phase angle is plotted against frequency. The peak has, in general, been observed to occur in the region of 0.05 to 0.1 rad/sec. These results substantiate the observations of Pagen and others who have studied the dynamic response of bituminous concrete in sinusoidal testing (15, 16, 17).

Form of the Transfer Function

Table 2 shows a set of typical transfer functions obtained for the specimens tested at 80 F, from which it is seen that the transfer function derived for each individual case

TABLE 1

RESULTS OF A TYPICAL SINUSOIDAL TEST OF SPECIMEN Z-1 AT TEST TEMPERATURE 80 F

ω	a_1	a_2	$R = \frac{a_2}{a_1}$	R_{db}	t	ϕ	
						rad	deg
0.025	1000	22.0	0.0220	-33.1520	15.30	0.382	21.8
0.050	1000	19.5	0.0195	-34.4240	8.32	0.416	23.9
0.100	1000	17.0	0.0170	-35.3920	4.15	0.415	23.8
0.250	1000	11.0	0.0110	-39.1720	1.50	0.375	21.5
0.500	1000	9.0	0.0090	-40.9160	0.68	0.340	19.5
1.000	1000	8.0	0.0080	-41.9380	0.311	0.311	17.8
2.500	1000	6.5	0.0065	-43.7420	0.110	0.271	15.5
5.000	1000	5.0	0.0050	-46.0200	0.046	0.230	13.2
10.000	1000	4.0	0.0040	-47.9580	0.021	0.210	12.0

Note: ω = input frequency, rad/sec,
a_1 = input magnitude, mv,
a_2 = output magnitude, mv,
R = a_2/a_1,
R_{db} = R expressed in decibels = $20 \log_{10} R$,
t = time lag between input and output, sec, and
ϕ = phase lag between input and output = ωt radians.

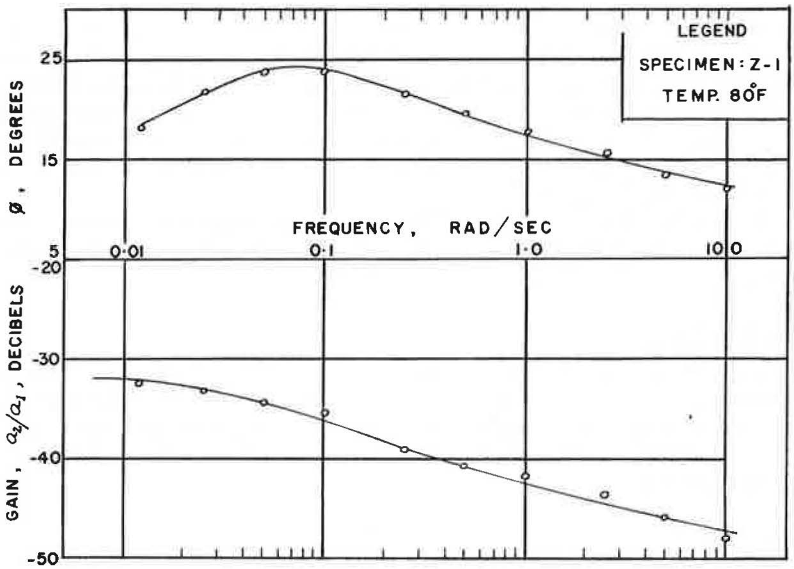

Figure 8. Typical frequency response curve.

is of the form

$$G(s) = \frac{A(s + a_1)(s + a_2)(s + a_3)}{(s + b_1)(s + b_2)(s + b_3)(s + b_4)} \tag{12}$$

where A is a constant and a_i's and b_i's are the corner frequencies previously defined and obtained in the process of approximating the frequency spectrum with asymptotes. The significant feature of this form of the transfer function is that the factors in both the numerator and denominator are to first power. It may be recalled that this is due to the geometric procedure followed in approximating the frequency spectrum with asymptotes of slopes of 6 decibels per octave. It was also mentioned that any slope can be used, in which case it is obvious from Eq. 10 that the transfer function may contain fraction powered or higher powered terms in either the numerator or the denominator or both. This form is not desirable because the resulting transfer function will be too difficult or even impossible to handle through Laplace transforms.

From the procedure followed in approximating the frequency spectrum with asymptotes, it is obvious that the number of factors in the numerator and denominator of the transfer function in Eq. 12 depends on the number of asymptotes used to approximate the frequency spectrum. It was found from the analysis of the experimental results that the transfer function obtained using 8 asymptotes was accurate enough for practical purposes.

TABLE 2

TRANSFER FUNCTIONS AT 80 F

Specimen	Transfer Function
z-1	$G(s) = \frac{(0.05)(s+0.025)(s+0.2)(s+2)}{(s+0.02)(s+0.1)(s+1)(s+10)}$
x-1	$G(s) = \frac{(0.0415)(s+0.028)(s+0.17)(s+1.8)}{(s+0.02)(s+0.1)(s+1)(s+10)}$
y-1	$G(s) = \frac{(0.04)(s+0.025)(s+0.16)(s+2.3)}{(s+0.02)(s+0.1)(s+1)(s+10)}$
z-2	$G(s) = \frac{(0.04)(s+0.025)(s+0.16)(s+2.3)}{(s+0.02)(s+0.1)(s+1)(s+10)}$
x-2	$G(s) = \frac{(0.0415)(s+0.028)(s+0.17)(s+1.8)}{(s+0.02)(s+0.1)(s+1)(s+10)}$
y-2	$G(s) = \frac{(0.0415)(s+0.028)(s+0.17)(s+1.8)}{(s+0.02)(s+0.1)(s+1)(s+10)}$

Effect of Mix Type on Transfer Function

The 2 different bituminous mixes studied in this investigation were quite dissimilar in their aggregate gradings and in their binder contents, namely, 5.0 and 3.5 percent by weight of aggregate for mix 1 and mix 2, respectively. However, it is apparent from Figure 9 that there was no appreciable difference in the transfer functions of the specimens, irrespective of their composition or orientation when tested at any one temperature. Even though the grading and asphalt content were different for the 2 mixes, their density-void ratio characteristics were similar, namely 142.6 lb/cu ft and 6.5 percent for mix 1 and 142.0 lb/cu ft and 8.5 percent for mix 2.

It can be anticipated that density will have a marked influence on the response to loading of a given mix and hence on its transfer function. However, more work is necessary to study the effect of this and other mix variables.

Effect of Specimen Orientation

It was pointed out earlier that for each mix 3 specimens were cut along 3 mutual perpendicular directions, designated x, y, and z. Figure 9 shows that there is not much difference in the transfer functions of the differently oriented specimens of either mix at any one temperature. This suggests that effect of anisotropy, if any, is not reflected in the transfer function.

Effect of Temperature

The experiments in the present investigation were conducted at 70, 80, and 90 F. By examining the absolute value curves in Figure 9, it is readily seen that temperature plays an important role in the transfer function of any specimen. Asphalt cement being thermoplastic, the asphaltic concrete specimen will yield increased displacements for increases in temperature at a given stress level. This results in a decrease in viscosity of the binder in the asphaltic concrete as temperature increases. Thus the output-input ratio for a constant input at any frequency will increase with decrease in viscosity or increase in temperature. This is precisely what is shown by the curves.

Examination of the transfer functions revealed that, for a given specimen, the various factors in the numerator and denominator do not seem to vary much with change in temperature, but that the coefficient A varies appreciably. It would be of interest to study this change in A with reference to the viscosity of asphalt cement or the asphalt cement-filler matrix.

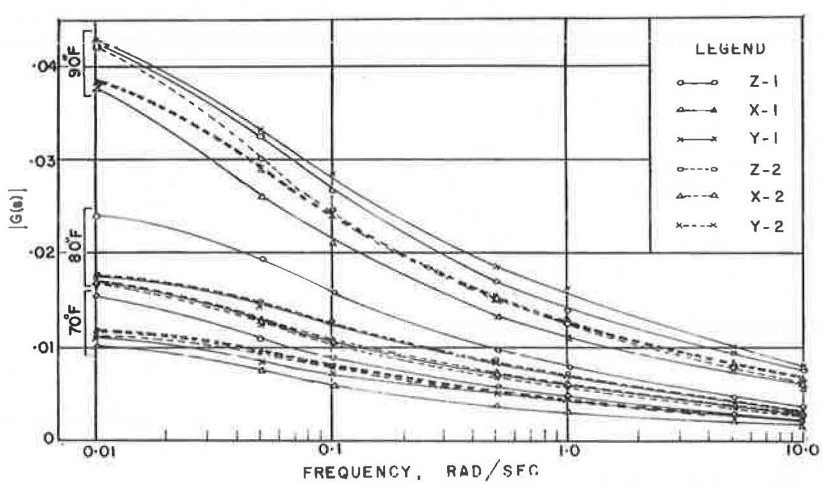

Figure 9. Curves of |G(s)| versus frequency.

Effect of Specimen Size on Transfer Function

Frequency-response tests were conducted on 3 different-sized specimens with constant height-width ratio using mix 1 at 90 F. They were cut along the same direction in the compacted beam. The large, medium, and small sizes were $1\frac{1}{4}$ by $1\frac{1}{4}$ by $2\frac{1}{2}$ in., 1 by 1 by 2 in., and $\frac{3}{4}$ by $\frac{3}{4}$ by $1\frac{1}{2}$ in. respectively. Analysis of their test results showed that the frequency spectrums for the 3 specimens are nearly the same and that they can be approximated by one single transfer function, which shows that the transfer function for the mix is independent of the specimen size.

In the light of this observation, it appears that the transfer function can be used to represent the dynamic characteristics of a viscoelastic material in much the same way as the elastic modulus represents the stress-strain characteristics of an elastic material. In other words, for a given viscoelastic material such as a bituminous concrete, there is only one transfer function for a given temperature.

Prediction of Displacements for Sinusoidal Loads

It was seen earlier that, once the transfer function of a system is known, it can be used to predict the output of the system for any other given input that is a function of time. After the transfer functions for the asphaltic concrete test specimens under investigation were determined, attempts were made to predict the output displacement of the specimens when subjected to a sinusoidal input of known magnitude, other than the one used to determine the transfer function. Each specimen was tested under 3 sinusoidal inputs of known magnitude, other than the one used to determine the transfer function. Each specimen was tested under 3 sinusoidal inputs at each of the 3 temperatures used previously.

From Eq. 3, the solution for the output displacement of a dynamic system with a transfer function G(s) for a sinusoidal load input of $f(t) = f_o \sin \omega t$ is given by

$$x(t) = f_o\,|G(s)|\,\sin(\omega t + \phi) \qquad (13)$$

where f_o = magnitude of the input,
ω = frequency of the input, and
ϕ = phase angle between output and input.

In Eq. 13, x(t) is a maximum when $\sin(\omega t + \phi) = 1$, so that

$$x(t)_{max} = f_o\,|G(s)| \qquad (14)$$

Experimentally, it is convenient to measure $x(t)_{max}$ at the peak of the sinusoidal displacement output. Thus, the calculated displacement from Eq. 14 can be compared with the measured displacement for a given magnitude and frequency.

It was mentioned elsewhere that in the experimental setup of this investigation, the input was recorded in pounds and millivolts and the output was recorded in inches and millivolts. Thus the output-input ratio can be either dimensionless or in units of inches per pound. The frequency spectrum and the transfer function analyses were obtained as dimensionless quantities. In applying units to Eq. 14, an experimental constant K needs to be introduced and it becomes

$$x(t)_{max} = K\,f_o\,|G(s)| \qquad (15)$$

From the sensitivity control values of the recorder, K was calculated and found to be

$$K = 25 \times 10^{-4} \text{ in./lb} \qquad (16)$$

The calculated and measured values for the maximum displacement for all the specimens tested at 80 F are given in Table 3. In all cases the magnitude of the sinusoidal

TABLE 3

CALCULATED AND MEASURED DISPLACEMENTS AT 80 F

Specimen	Sine Load Input				Constant Load Input		
			x(t)			x(t)	
	f_o (lb)	ω (rad/sec)	Calculated (in. x 10^{-4})	Measured (in. x 10^{-4})	f_o (lb)	Calculated (in. x 10^{-4})	Measured (in. x 10^{-4})
z-1	5	0.05	2.37	2.25	5	3.13	2.50
	5	0.50	1.20	1.25	10	6.26	5.00
	5	5.00	0.58	0.50	15	9.39	10.00
x-1	5	0.05	1.59	1.25	5	2.23	2.00
	5	0.50	0.87	0.75	10	4.46	3.75
	5	5.00	0.50	0.50	15	6.69	5.50
y-1	15	0.05	5.32	5.00	5	2.30	2.25
	15	0.50	3.18	3.00	10	4.60	4.00
	15	5.00	1.43	1.75	15	6.90	8.75
z-2	15	0.05	5.32	5.00	5	2.30	2.25
	15	0.50	3.18	3.00	10	4.60	4.50
	15	5.00	1.43	1.50	15	6.90	6.50
x-2	15	0.05	4.76	5.00	5	2.23	1.25
	15	0.50	2.62	2.50	10	4.46	4.00
	15	5.00	1.50	1.75	15	6.69	9.50
y-2	15	0.05	4.76	4.75	5	2.23	2.25
	15	0.50	2.62	2.50	10	4.46	4.00
	15	5.00	1.50	1.50	15	6.69	7.00

input was kept the same and only the frequency was varied. Comparison of the calculated and the measured displacement values clearly indicates the close agreement between these values in all cases. Besides proving the efficacy of the transfer function as a displacement predicting tool for viscoelastic materials, this also shows that the bituminous concrete behaves as a linear system at the levels of stress considered in the tests.

Prediction of Displacements for Static Loads

The transfer function can be used to predict the time-dependent displacements of a system subjected to a step function input, and, mathematically, a static load can be represented by a step function. For the transfer function given by Eq. 12, the solution for the displacement under a static load can be shown as

$$x(t) = C_1e^{b_1t} + C_2e^{b_2t} + C_3e^{b_3t} + C_4e^{b_4t} + C_5 \tag{17}$$

where C_i is a constant independent of t for the system and b_i is the root of the denominator of the transfer function that is observed to be negative in the case of the experimentally derived transfer functions.

Two boundary conditions are applicable to Eq. 17: (a) when t = 0 and (b) when t = ∞. When t = 0,

$$x(t) = 0 \tag{18}$$

that is

$$C_5 = C_1 + C_2 + C_3 + C_4 \tag{19}$$

When t = ∞,

$$x(t) = C_5 \tag{20}$$

When a constant load is applied to a viscoelastic material, the displacement approaches a constant value after a certain time depending on the nature of the material. In the laboratory, for the bituminous concrete specimens tested, this time was observed to be in the order of a few minutes. Hence, the second boundary condition can be applied to this steady displacement value in the test; that is, Eq. 20 can be used to calculate the displacement. The limitations of using Eq. 20 are recognized in that in a test t does not reach infinity and hence the measured value will be slightly lower than the calculated value.

In order to get the calculated displacement in proper units, Eq. 20 should be multiplied by the experimental constant K, described in the previous section so that

$$x(t) = K\,C_5 \tag{21}$$

where $K = 25 \times 10^{-4}$ in./lb. C_5 has the units of pound.

The calculated and measured values of the displacement at 80 F for the specimens, each under 3 different constant loads, are given in Table 3. Comparison of the calculated and the measured displacement values clearly indicates the close agreement between the two in almost all cases. This observation is very significant and it brings to focus the following points:

1. The mathematical theory of transfer functions is applicable to viscoelastic materials in general and to bituminous concrete in particular;
2. The techniques developed to derive the transfer function in this investigation are sound; and
3. The transfer function serves as a connecting link between the responses of the material tested under dynamic and static loads.

Differential Equation From Transfer Function

Once the transfer function is known for a given system, the differential equation for the system behavior can be written using the Laplace inverse transform. For bituminous concrete, the transfer function was seen to be of form

$$G(s) = \frac{A(s + a_1)(s + a_2)(s + a_3)}{(s + b_1)(s + b_2)(s + b_3)(s + b_4)} \tag{12}$$

From definition

$$G(s) = \frac{\bar{x}(s)}{\bar{f}(s)} \tag{1}$$

From Eqs. 1 and 12,

$$\frac{\bar{x}(s)}{\bar{f}(s)} = \frac{A(s + a_1)(s + a_2)(s + a_3)}{(s + b_1)(s + b_2)(s + b_3)(s + b_4)} \tag{22}$$

Rewriting,

$$\bar{x}(s)(s + b_1)(s + b_2)(s + b_3)(s + b_4) = A\,\bar{f}(s)(s + a_1)(s + a_2)(s + a_3) \tag{23}$$

Multiplying and rearranging the terms,

$$\bar{x}(s)\left[s^4 + B_1 s^3 + B_2 s^2 + B_3 s + B_4\right] = A\,\bar{f}(s)\left[s^3 + B_5 s^2 + B_6 s + B_7\right] \tag{24}$$

where $B_1 = b_1 + b_2 + b_3 + b_4,$ (25)
$B_2 = b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1 + b_4b_2 + b_3b_1,$ (26)
$B_3 = b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_1 + b_4b_1b_2,$ (27)
$B_4 = b_1b_2b_3b_4,$ (28)
$B_5 = a_1 + a_2 + a_3,$ (29)
$B_6 = a_1a_2 + a_2a_3 + a_3a_1,$ and (30)
$B_7 = a_1a_2a_3.$ (31)

Rewriting Eq. 24,

$$s^4\bar{x}(s) + B_1s^3\bar{x}(s) + B_2s^2\bar{x}(s) + B_3s\bar{x}(s) + B_4\bar{x}(s)$$

$$= A\left[s^3\bar{f}(s) + B_5s^2\bar{f}(s) + B_6s\bar{f}(s) + B_7\bar{f}(s)\right] \tag{32}$$

Applying the Laplace inverse transform,

$$\frac{d^4x(t)}{dt^4} + B_1\frac{d^3x(t)}{dt^3} + B_2\frac{d^2x(t)}{dt^2} + B_3\frac{dx(t)}{dt} + B_4x(t)$$

$$= A\left[\frac{d^3f(t)}{dt^3} + B_5\frac{d^2f(t)}{dt^2} + B_6\frac{df(t)}{dt} + B_7f(t)\right] \tag{33}$$

Equation 33 is a fourth order linear differential equation that has constant coefficients and that describes the dynamics of the bituminous concrete. The input (force) and the output (displacement) are the equation's 2 variables that are functions of time. The constants, B_i's, are easily determined from the roots of the denominator, b_i's, and from those of the numerator, a_i's, of the transfer function, using Eqs. 25 through 31.

It is of significance to note here that these roots of the differential equation are obtained as the corner frequencies directly from the frequency spectrum for the material under test, and no assumption whatsoever has been made in deriving the equation.

CONCLUSIONS

Based on the results and within the limitations of this investigation, the following conclusions are enumerated:

1. The viscoelastic or time-dependent characteristics of an asphaltic concrete can be represented by a transfer function G(s) that is a function of the complex variable s. The transfer function is unique for the material at a given temperature. It is possible to obtain this function experimentally from a series of sinusoidal load tests on the material.
2. The transfer function for an asphaltic concrete is of the form

$$G(s) = \frac{A(s + a_1)(s + a_2)(s + a_3)}{(s + b_1)(s + b_2)(s + b_3)(s + b_4)} \tag{12}$$

where A is a constant, and a_i's and b_i's are roots of the numerator and denominator respectively.

3. The roots of the denominator of G(s), namely b_1, b_2, b_3, and b_4, are negative, real, and distinct, thus indicating that bituminous concrete behaves as an overdamped system.
4. Parameters obtained in the transfer function for bituminous concrete are believed to be better indicators of material performance than those commonly used, such as Young's modulus and Poisson's ratio, which change with rate and time of loading.

5. Temperature is the one single factor that has the greatest effect on the transfer function of asphaltic concrete. Increase in temperature increases the value of the constant A in the transfer function equation.

6. The transfer function is a powerful tool useful in predicting the displacement of asphaltic concrete under an applied load, dynamic or static. By treating the static load as a step function of time, the resulting displacement can be calculated by means of the transfer function derived experimentally from the dynamic test. The excellent agreement between the calculated and measured values of the displacement in this investigation validates the concept that the transfer function represents a material property that is independent of the type of load input.

7. Through the use of the transfer function and without assuming any spring-dashpot model, it is possible to represent the time-dependent behavior of asphaltic concrete by a fourth order linear differential equation with constant coefficients. The coefficients can be computed from the roots of the denominator and numerator of the transfer function.

ACKNOWLEDGMENTS

The authors acknowledge the assistance of the Bureau of Public Roads, Federal Highway Administration, U. S. Department of Transportation, who supported the investigation with the Indiana State Highway Commission. Laboratory testing and experimental techniques were carried out in the Bituminous Materials Laboratory of the Joint Highway Research Project at Purdue University, Lafayette, Indiana.

REFERENCES

1. Lee, E. H. Stress Analysis in Viscoelastic Bodies. Quarterly of Applied Mathematics, July 1955.
2. Biot, M. A. Dynamics of Viscoelastic Anisotropic Media. Proc., Fourth Midwestern Conference on Solid Mechanics, Sept. 1955.
3. Biot, M. A. Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena. Jour. of Applied Physics, Vol. 25, No. 11, Nov. 1954.
4. Lee, E. H. Viscoelastic Stress Analysis. In Structural Mechanics, Proc., First Symposium on Naval Structural Mechanics, Pergamon Press, New York, 1960.
5. Lee, E. H., and Rogers, T. G. Solution of Viscoelastic Analysis Problems Using Measured Creep or Relaxation Functions. Jour. of Applied Mechanics, Vol. 30, No. 1, March 1963.
6. Pister, K. S., and Monismith, C. L. Analysis of Viscoelastic Flexible Pavements. HRB Bull. 269, 1960, pp. 1-15.
7. Pister, K. S. Viscoelastic Plate on a Viscoelastic Foundation. Jour. of Engineering Mechanics Div., ASCE, Vol. 87, Proc. Paper EM1, Feb. 1961.
8. Pister, K. S., and Williams, M. L. Bending of Plates on a Viscoelastic Foundation. Jour. of Engineering Mechanics Div., ASCE, Vol. 86, Proc. Paper EM5, Oct. 1960.
9. Reissner, E. A Note on Deflections of Plates on Viscoelastic Foundation. Trans., ASME, Vol. 25, No. 1, March 1958.
10. Hoskin, B. C., and Lee, E. H. Flexible Surfaces on Viscoelastic Subgrades. Jour. of Engineering Mechanics Div., ASCE, Vol. 85, Proc. Paper EM4, Oct. 1959.
11. Harr, M. E. Influence of Vehicle Speed on Pavement Deflection. HRB Proc., Vol. 41, 1962, pp. 77-82.
12. Wood, L. E., and Goetz, W. H. Rheological Characteristics of a Sand-Asphalt Mixture. Proc., Assn. Asphalt Paving Technologists, Vol. 28, 1959.
13. Secor, K. E., and Monismith, C. L. Analysis of Triaxial Test Data on Asphalt Concrete Using Viscoelastic Principles. HRB Proc., Vol. 40, 1961, pp. 295-314.
14. Kuhn, S. H., and Rigden, P. J. Measurement of Viscoelastic Properties of Bitumens Under Dynamic Loading. HRB Proc., Vol. 38, 1959, pp. 431-463.
15. Papazian, H. S. The Response of Linear Viscoelastic Materials in the Frequency Domain with Emphasis on Asphaltic Concrete. Proc. , Internat. Conference on the Structural Design of Asphalt Pavements, Univ. of Michigan, Ann Arbor, 1962.

16. Pagen, C. A. Rheological Response of Bituminous Concrete. Highway Research Record 67, 1965, pp. 1-26.
17. Kallas, B. F., and Riley, J. C. Mechanical Properties of Asphalt Pavement Materials. Proc., Second Internat. Conference on the Structural Design of Asphalt Pavements, Univ. of Michigan, Ann Arbor, 1967.
18. Truxal, J. C. Control System Synthesis. McGraw-Hill, New York, 1955.
19. Nixon, F. E. Principles of Automatic Controls. Prentice Hall, 1960.
20. Goldberg, J. H. Automatic Controls: Principles of Systems Dynamics. Allyn and Bacon, Boston, 1964.
21. Tsien, H. S. Engineering Cybernetics. McGraw-Hill, New York, 1954.
22. Etkin, B. Dynamics of Flight. John Wiley and Sons, New York, 1959.
23. Crafton, P. A. Shock and Vibration in Linear Systems. Harper and Brothers, 1961.
24. Busching, H. W., Goetz, W. H., and Harr, M. E. Stress-Deformation Behavior of Anisotropic Bituminous Mixtures. Proc., Assn. Asphalt Paving Technologists, Vol. 36, 1967.
25. Lister, N. W., and Jones, R. The Behavior of Flexible Pavements Under Moving Wheel Loads. Proc., Second Internat. Conference on the Structural Design of Asphalt Pavements, Univ. of Michigan, Ann Arbor, 1967.
26. Swami, S. A. The Response of Bituminous Mixtures to Dynamic and Static Loads Using Transfer Functions. Purdue Univ., Lafayette, Indiana, PhD dissertation, Jan. 1969.