

Characteristics of the Elastic Deformations of Concrete

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The instantaneous deformations of concrete are explained as a result of elastic deformations and internal crack propagation. Empirical formulas are presented for the numerical approximation of the stress-strain diagram. The definitions and various static and dynamic test methods are discussed for the determinations of the Young's modulus, shear modulus, and Poisson's ratio. Related formulas are also presented. Comparisons are made between static Young's moduli determined by various methods, between dynamic Young's moduli obtained by various methods, between static and related dynamic moduli, and between static and dynamic Poisson's ratios. It is shown that the constants determined by dynamic methods are greater than the corresponding static constants. Attempt is made to explain the revealed differences. Data concerning the reproducibility of the elastic constants close the paper indicating that the test methods for dynamic constants are superior in this respect. The static elastic constants of concrete are of greater significance to the design engineer than the corresponding dynamic values. On the other hand, the determination of the dynamic constants is easier.

•WHEN A LOAD is applied to a body, the body is deformed. The amount and character of deformation depend on the properties of the body, the particular environment, the magnitude of the load, the rate at which it is applied, and the elapsed time after the load application that the observation is made. Different materials vary widely in their response to load. This response is called rheological behavior. Although instantaneous response and time-depending response are not entirely separable, it is convenient to consider them separately as elastic deformations and plastic deformations respectively (1). A deformation is said to be elastic if it appears and disappears immediately on application and removal of stress.

A knowledge of the deformability of concrete is necessary to compute stresses from observed strains, to proportion sections of highway slabs and reinforced concrete members when certain design procedures are used, and to compute loss of prestress in prestressed structures. In the study of models of concrete structures, it is also necessary to know the stress-strain characteristics of the material for the model so that dimensional similarity may be obtained.

SHAPE OF THE STRESS-STRAIN DIAGRAM FOR CONCRETE

The instantaneous deformations of a specimen under load can be described conveniently by a stress-strain diagram. Figure 1 shows such a typical diagram for the axial deformation of a concrete specimen loaded with 2 different rates in uniaxial compression or tension. The stresses are averages (loads divided by cross-sectional area) and so too are the strains. The diagram is influenced considerably by the testing conditions, such as type of testing machine, rate of loading, size and shape of specimen, size and location of strain gages, and number of load repetitions (2, 3, 4, 5), as well as by the age and composition of concrete, such as type and quantity of aggregate, and especially the porosity (6), in a similar but not identical way as the concrete strength is influenced

by most of these factors (7, 8). Torque-twist diagrams for concrete are similar to the stress-strain diagram (9) and serve a similar purpose.

Along with the axial deformations, transverse deformations also take place under load. It is customary to characterize the lateral deformation by the Poisson's ratio, which is the absolute value of the ratio of transverse strain to the corresponding axial strain resulting from a uniformly distributed axial stress below the proportional limit of that material.

The shape of the stress-strain diagram or the torque-twist diagram or the stress-Poisson's ratio diagram for concrete can be explained in qualitative terms from the well-established fact that the fracture of concrete takes place through progressive internal cracking (10). The compressive stress-strain diagram starts out with a nearly linear portion that stretches to about 30 percent of the ultimate load. Beyond this point the curve deviates gradually from the straight line. This is attributed chiefly to the increase of cracks on the interface of coarse aggregate and mortar, that is, to the start of discontinuity in the specimen and later to extensive mortar cracking, although some creep probably also takes place (11, 12). In accordance with this mechanism, the stress-strain diagram is usually more curved for concrete than for the comparable mortar that usually has a less flat curve than the comparable paste. This failure mechanism also implies that (a) the stress-strain diagram of concrete becomes steeper and more nearly linear with the increase of the rate of loading; (b) the curvature of the diagram increases with the amount and size of the aggregate in the concrete; (c) the shape of the diagram obtained by tensile load is similar to, although more nearly linear than, the shape of the comparable diagram obtained by compressive load; and (d) the major part of the concrete deformations that is caused by the internal cracking and creep is permanent. Experimental evidence supports these implications (10). Also, when concrete and mortar specimens are subjected to increasing uniaxial compression, the Poisson's ratio for each begins to increase on attaining a certain stress level. After that the volume of the specimen also begins to increase. This can be explained again by the appearance and propagation of cracking, because hardened paste specimens continue to consolidate at an increasing rate with increased load, and stone specimens show only a slight volume expansion at stresses near failure (13).

NUMERICAL APPROXIMATIONS FOR THE STRESS-STRAIN DIAGRAM OF CONCRETE

Despite the supporting experimental evidence, the best that the theory of internal cracking can do is to describe the stress-strain relationship in qualitative terms. Other theories are even less suitable for this purpose. Thus, only empirical formulas obtained from boundary conditions and by curve-fitting are available for the numerical approximation of the stress-strain diagram of concrete. Several of these formulas are compared, as shown in Figure 2, in terms of f/f_0 relative stress and ϵ/ϵ_0 relative strain. The limits of validity and degrees of approximation of these formulas are restricted. Further analysis of these formulas is presented elsewhere (7).

THE STATIC ELASTIC CONSTANTS OF CONCRETE

The most common measure of elasticity is the modulus of elasticity. This term is defined, in general, by the Standard Definitions of Terms Relating to Methods of Me-

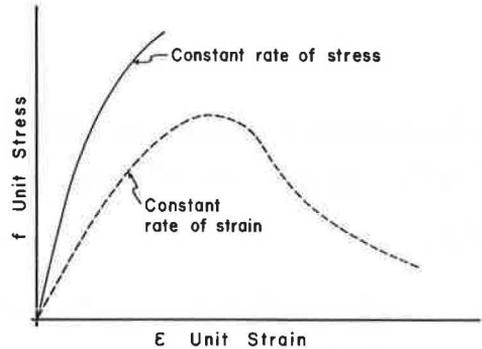


Figure 1. Two typical stress-strain curves for concrete under uniaxial load. The top curve is characteristic of a loading process where the rate of stress increase is kept constant during the testing. The bottom curve is obtained by keeping the rate of strain increase constant.

chanical Testing (ASTM Designation E 6-66) as the ratio of stress to corresponding strain below the proportional limit. Modulus of elasticity, like stress, is expressed in force per unit of area (psi or kg/cm²). For concrete, as well as for other materials where the stress-strain relationship is curvilinear rather than linear, one of the 4 following terms may be used (Fig. 3):

1. Initial tangent modulus—The slope of the stress-strain curve at the origin;
2. Tangent modulus—The slope of the stress-strain curve at any specified stress or strain;
3. Secant modulus—The slope of the secant drawn from the origin to any specified point on the stress-strain curve; and
4. Chord modulus—The slope of the chord drawn between any 2 specified points on the stress-strain curve.

All 4 definitions refer to the static modulus of elasticity in contrast to the modulus of elasticity computed from dynamic tests.

Static modulus of elasticity may be measured in compression, tension, or shear. The modulus in tension or compression is frequently referred to as Young's modulus of elasticity and is designated by E. The shear modulus is sometimes called the modulus of rigidity or torsional modulus and is designated by G. These 2 moduli describe the elastic behavior of a homogeneous and isotropic material completely. Thus, the third usual elastic constant, the Poisson's ratio, can already be expressed in terms of E and G, as follows:

$$\mu = \frac{E}{2G} - 1 \quad (1)$$

where μ is the Poisson's ratio.

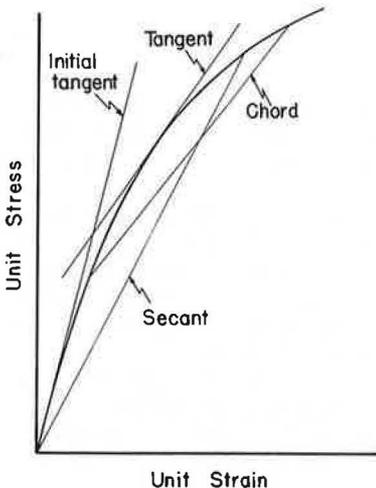


Figure 3. Four definitions for the static Young's modulus of elasticity of concrete.

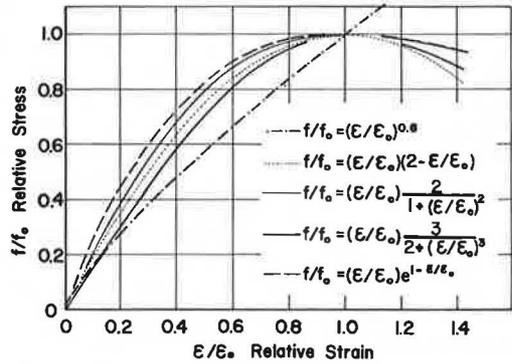


Figure 2. Comparison of several formulas for the stress-strain diagram of concrete.

The only ASTM standard method for the determination of static elastic constants of concrete is the Standard Method of Test for Static Young's Modulus of Elasticity and Poisson's Ratio in Compression of Cylindrical Concrete Specimens (ASTM Designation C 469-65). It stipulates a chord modulus between 2 points on the stress-strain curve defined as follows: The lower point corresponds to a strain of 50 micro-inches/inch (μ in./in.), and the upper point corresponds to a stress equal to 40 percent of the strength of concrete at the time of loading. The lower point is near the origin but far enough removed from the origin to be free of possible irregularities in strain reading caused by seating of the testing machine platens and strain measuring devices. The upper point is taken near the upper end of the working stress range that was assumed in design. Thus, the determined modulus is approximately the average modulus of elasticity in compression throughout the working stress range (1). The specimen should be loaded at least twice without recording any data during the first loading. Calculations should be based on the average of the results of subsequent loadings. At least 2 subsequent loadings are recommended.

Usually cylinders or prisms are used for the determination of the modulus of elasticity in compression. In order to compensate for the effect of eccentric loading or non-uniform response by the specimen, strains should be measured along the axis of the specimen or along 2 or more gage lines uniformly spaced around the periphery of the cylinder. The selection of the gage length is important. It must be large in comparison with the maximum particle size of the aggregate so that local strain irregularities do not unduly influence the results. It must not, however, encroach on the ends of the specimen because strains near the ends may differ from strains elsewhere. Half the specimen height is a good gage length. Both mechanical devices and electrical gages have been used successfully for strain measurements of concrete.

The Young's modulus can also be determined, at least theoretically, on specimens loaded as beams. Deflection measurements, curvature measurements, or strain measurements in the extreme fibers can be used for this purpose. The usual approach is to measure deflections caused by known loads and to calculate the modulus of elasticity from a suitable beam-deflection formula. Unfortunately, the application of the usual simple deflection formulas provide unreliable E values for concrete for 2 reasons. First, these formulas are based on the assumption that the material follows Hooke's law, which in the case of concrete is not true. Second, the depth-to-span ratios of unreinforced concrete beams normally used for such tests are so large that shear deflection composes a significant part of the total deflection. A beam-deflection formula for center-point loading and corrected for shear (1, 14) is as follows:

$$E_{st} = \frac{Pl^3}{48FI} \left[1 + (2.4 + 1.5\mu)(h/l)^2 - 0.84(h/l)^3 \right] \quad (2)$$

where

- E_{st} = static Young's modulus of elasticity;
- F = maximum deflection,
- P = applied central load,
- l = distance between supports,
- I = moment of inertia of the section,
- μ = Poisson's ratio, and
- h = depth of the beam.

When the beam has a 6- by 6-in. cross section and is loaded at the third-points of an 18-in. span, the following formula can be used (15):

$$E_{st} = \frac{P}{F} (1.16 + 0.125\mu) \quad (3)$$

where the P total load is expressed in pounds, F in inches, and E_{st} in psi. Equations 2 and 3 are not corrected for the deviation from Hooke's law. Thus, the values they provide for the E_{st} of a concrete are greater than the actual values, often to a considerable degree.

Static determination of Poisson's ratio is made by measuring both the axial strain and the corresponding transverse strain. Details are described by Yoshida (16). A standard test method is given in ASTM C 469. Poisson's ratio is also commonly computed by Eq. 1 from results of Young's modulus and shear modulus. The shear modulus is most often determined dynamically or by calculation from E and μ . When a direct static determination is desired, a torsion test is the common procedure.

Static modulus of elasticity and Poisson's ratio can be calculated from splitting tensile test if principal strains are measured at the center of the concrete cylinder (17). In this case, the gage length should be limited to about 0.07 of the specimen diameter because the strains encountered in this test are of rapidly changing character. Also, this method is sensitive to observational errors.

Static tests for Young's modulus and Poisson's ratio involve the application of stresses of the same order as those in practice, whereas the stresses induced in the dynamic tests are very small. Because these elastic constants are stress-dependent, the static

modulus and ratio are of greater significance to the design engineer than the corresponding dynamic values. On the other hand, the determination of the dynamic values is, in most cases, easier and more precise.

THE DYNAMIC ELASTIC CONSTANTS

The dynamic Young's modulus or the dynamic shear modulus is usually determined by the resonance frequency method. Rayleigh was the first to set forth simplified relationships among the resonant frequency of vibration of a homogeneous specimen, the velocity of a vibratory wave passing through the material, and the modulus of elasticity of the material (18). The first application of this method on concrete was published by Powers in 1938 (19). The ASTM standard forms of this method are based essentially on the more exact theory developed by Pickett (20) and are described in the Standard Method of Test for Fundamental Transverse, Longitudinal, and Torsional Frequencies of Concrete Specimens (ASTM Designation C 215-60). The recommended test specimens are beams that are made in accordance with the standard procedure prescribed for the flexural specimens, but other suitable specimens, such as cylinders, can also be used. The elastic constants may be calculated from the following formulas:

$$E_R = CWn^2 \quad (4)$$

$$E_R = DW(n')^2 \quad (5)$$

$$G_R = BW(n'')^2 \quad (6)$$

where

E_R = dynamic Young's modulus calculated from the fundamental resonance frequency of the concrete specimen;

G_R = dynamic shear modulus calculated from the fundamental torsional frequency;

\bar{W} = weight of the specimen;

n' = fundamental transverse frequency, cps; and

n'' = fundamental torsional frequency, cps.

The values of the C, D, and B factors depend on the shape and size of the specimen tested, on the units used, and on the Poisson's ratio of the concrete. Detailed instructions for the calculation of these factors may be found in ASTM Method C 215 or in the literature (21, 22, 23, 24, 25, 26). Equation 5 can be written in the following form when E_R is in psi and when the specimen is a rectangular prism:

$$E_R = 6 \times 10^{-6} L^2 d_1 (n')^2 \quad (7)$$

where

L = length of the prism, in.; and

d_1 = unit weight of the concrete, lb/cu ft.

Poisson's ratio does not enter heavily into the computations when Eq. 5 or Eq. 7 is used. Because the true value of Poisson's ratio is rarely known, it has become popular to use the longitudinal resonance technique for E_R so that the effect of an error in the assumed value of Poisson's ratio may be minimized. Longitudinal resonances of bars of square cross section or cylindrical rods provide particularly accurate results for Young's modulus (25). This test method is also described in the British Standard 1881.

The application of pulse transmission, which apparently was started on concrete by Long and Kurtz (27), is not recommended for the determination of the dynamic modulus of elasticity (the pulse modulus) by the pertinent ASTM standard Tentative Method of Test for Pulse Velocity Through Concrete (ASTM Designation C 597-67 T). The reasons for this may be the marked effect of Poisson's ratio on the pulse modulus as well as the fact that the usually applied calculations do not provide proper values for the pulse modulus if the cement paste and aggregate differ greatly in elastic properties (28, 29). If

this modulus is still to be computed from the pulse velocity, Whitehurst advises the use of the formula developed originally by Poisson for homogeneous materials as follows (21, 30):

$$E_p = \frac{V^2 d}{g} \frac{(1 + \mu)(1 - 2\mu)}{(1 - \mu)} \quad (8)$$

where

- E_p = dynamic Young's modulus calculated from the pulse velocity,
- V = longitudinal (compression) wave velocity,
- d = density, and
- g = gravitational acceleration.

If E_p is expressed in psi, V in ft/sec, $d = d_1$ in lb/cu ft, and g in ft/sec², the right side of Eq. 8 should also be divided by 144. Then a simplified form of this formula ($\mu = 0.24$) is as follows:

$$E_p = 0.000183d_1 V^2 \quad (9)$$

The V term in Eqs. 8 and 9 is actually the velocity of the wave front and is equivalent to the wave velocity only as long as the latter does not change with frequency. Because this appears to be true for concrete, the pulse velocity can be used in Eqs. 8 and 9 with little or no error (31).

Simultaneously with the longitudinal waves, transverse (shear) and surface (Rayleigh) waves are also generated by an impulse. The 3 types of wave travel with different velocities and become separated in time as the path length increases. The S , shear, and R , Rayleigh, wave velocities and their relationship to the elastic constants of a material (16, 22, 32, 33, 34) are

$$S^2 = \frac{G}{d} = \frac{gE}{2d(1 + \mu)} \quad (10)$$

and

$$R = pS \quad (11)$$

where $p = 0.911$ and 0.928 when $\mu = 0.2$ and 0.3 respectively.

It is possible, at least theoretically, to calculate the dynamic Poisson's ratio directly when, for instance, both the longitudinal frequency and the longitudinal wave velocity are available by matching Eqs. 5 and 8, or by using Eqs. 8 and 10 when both the longitudinal wave velocity and the shear wave velocity are measured, or when both the V and R velocities are available (16, 32, 35). The practical difficulty with these methods is that the S or R velocities or both cannot usually be determined accurately enough. The application of Eqs. 4 through 11 for concrete represents a more or less tolerable oversimplification because concrete is far from being a perfectly elastic, homogeneous, and isotropic material.

ESTIMATION OF THE ELASTIC CONSTANTS BY MODELS

Calculating the deformability of a composite solid material, such as concrete, given the deformabilities and amounts of the composing phases, is an important problem both from theoretical and practical standpoints. Such calculations are hindered by the fact that concrete has a complicated internal structure. Therefore, repeated attempts have been made to apply suitable simplified hypothetical structures, the so-called composite models, for such calculations.

Any model approach for composite materials involves considerable simplifying assumptions. In case of deformations, these assumptions concern stresses and strains as well as the relationship between them; for instance, elastic behavior of the model

components is usually assumed to be linear. More sophisticated models may provide more general applicability, but it is important to realize that every model contains a number of simplifying assumptions. Therefore, one should make the selection of the proper model for a given purpose by balancing the easy applicability of the model against a reasonable generality, considering that the increase of complexity of a model has usually a diminishing rate of return in reliability. It appears that the simplest models for the estimation of the modulus of elasticity, which still have an adequate reliability for considerable number of practical cases (36, 37), are represented by the following formulas:

(a) For normal-weight concrete,

$$\frac{1}{E_C} = 0.5 \left(\frac{1-g}{E_m} + \frac{G}{E_p} \right) + 0.5 \frac{1}{(1-g)E_m + gE_p} \quad (12)$$

(b) For lightweight-aggregate concrete,

$$E_L = \frac{0.5}{(1-g)/E_m + g/E_p} + 0.5 [(1-g)E_m + gE_p] \quad (13)$$

(c) For the effect of porosity,

$$E_p = 10^{-nv} E_0 \quad (14)$$

where

E_C , E_L , and E_p = estimated modulus of elasticity of normal-weight, lightweight, and porous concrete respectively;

E_0 = modulus of elasticity of the concrete with zero porosity;

E_m and E_p = modulus of elasticity of the cement paste and aggregate respectively;

g = fractional volume of aggregate (percent/100);

n = experimental constant; and

v = pore content.

All models are based on certain simplifying assumptions; consequently, their reliability and limits of validity are restricted. Thus, the model approach is best used for qualitative examination of material behavior.

COMPARISON OF THE VARIOUS KINDS OF YOUNG'S MODULI

The Young's moduli of a concrete obtained in various manners are usually more or less different. The size of this difference depends on several factors as described in the following sections.

Static Moduli of Elasticity

Walker (38) demonstrates that the relation of the tangent modulus to the secant modulus at the same load is 0.8 to 0.9 within practical stress limits. Davis and Troxell (39) find that the secant modulus of a granite concrete of about 2,500 psi compressive strength decreases almost linearly from 2.5×10^6 psi to 1.7×10^6 psi at the age of 28 days as the applied stress is increased from 200 to 1,000 psi. Data by Jones and Richart (40) show that the secant modulus of elasticity is about 10^5 psi less at $0.9f'_c$ than at $0.5f'_c$. Shideler (41) reports that values of the secant modulus of structural lightweight and normal concretes obtained at $0.3f'_c$ are almost identical with values obtained at $0.45f'_c$. Klieger's pertinent results as reported by Philleo (28) show a similar tendency.

Data by Klieger (15) demonstrate that the static modulus of elasticity determined by the compression of 6- by 12-in. cylinders is about three-fourths of the comparable

modulus obtained by the flexure of 6- by 6- by 30-in. beams on an 18-in. span and using Eq. 3. On the other hand, Witte and Price (42) report that the secant moduli determined by compression at the ages of 2 and 3 years are practically the same as the comparable moduli obtained by the flexure of 3- by 3- by 16 $\frac{1}{4}$ -in. beams and using, presumably, a simple deflection formula without any correction for shear or curvilinear stress distribution. Measurements by Vile show a practical equality of elastic modulus values obtained under uniaxial compressive and tensile states of stress (43). The differences between the initial tangent and secant moduli or between secant moduli at differing stresses or between tangent and chord moduli are due to the curvilinear character of the stress-strain diagram of concrete (Fig. 3). The discrepancies between compressive and flexural moduli can be attributed to the shear and the nonlinear stress distribution in flexure.

Dynamic Moduli of Elasticity

As Figure 4 shows, the moduli of elasticity calculated from longitudinal resonance frequencies and those from transverse frequencies are practically identical (44, 45). Data by Obert and Duvall (46) support this finding. The agreement between the resonance modulus and the modulus calculated from the pulse velocity is less satisfactory. As a rule, the E_p pulse moduli are greater. This is shown in Figure 5 where pertinent values obtained on the same specimens that provided data for Figure 4 are plotted (44, 45). Data published by other investigators (32, 47) likewise show that E_p is about 10 percent greater than E_r of the same concrete. Similarly, according to Philleo (28), Whitehurst obtained pulse moduli that were up to 47 percent greater than the corresponding resonance moduli, with an average difference of 15 percent. Philleo's own data indicate a similar tendency.

Klieger (15) determined the moduli of elasticity of a series of concrete mixtures by the resonance method both on 6- by 12-in. cylinders and on comparable 6- by 6- by

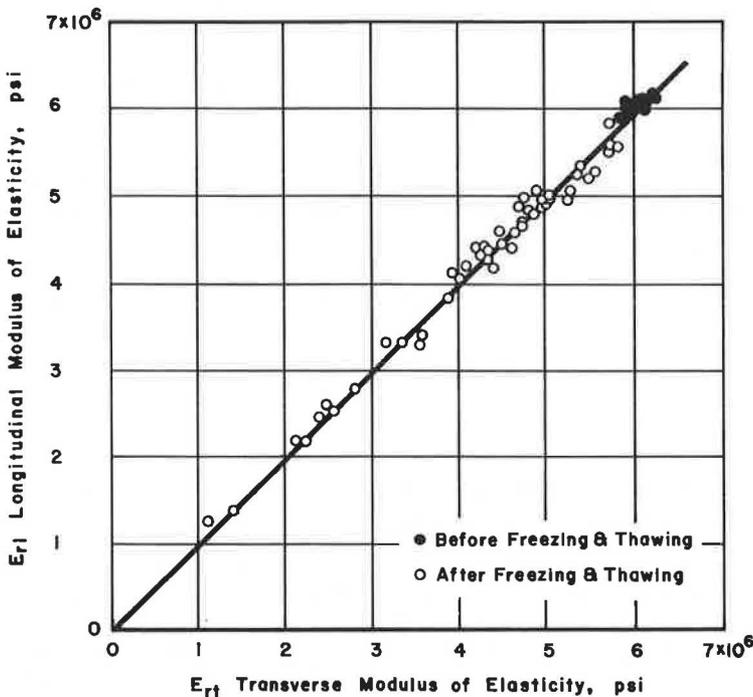


Figure 4. Comparison of moduli of elasticity from transverse resonance frequencies to moduli from longitudinal resonance frequencies (44, 45).

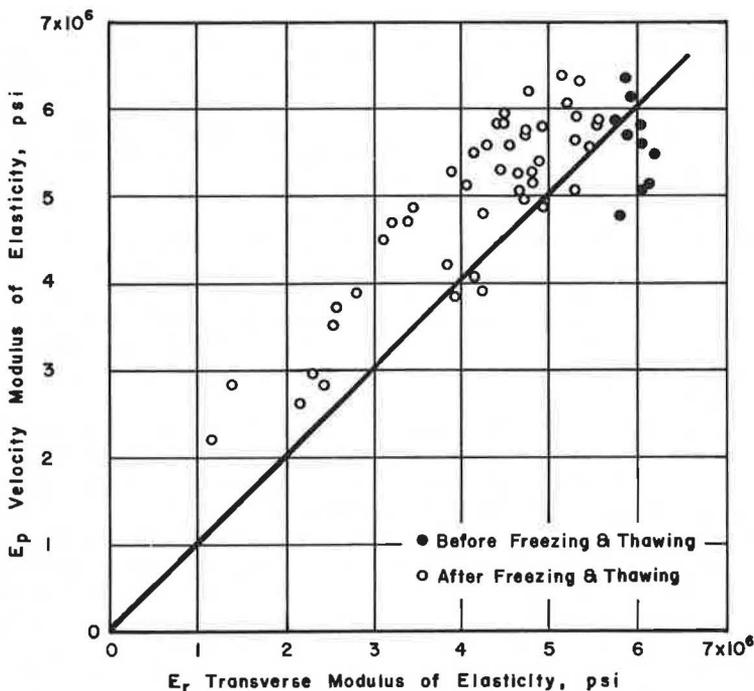


Figure 5. Comparison of moduli of elasticity from transverse resonance frequencies to moduli from pulse velocity (44, 45).

30-in. beams, having found the latter values to be about 8 percent higher. A similar test series by Stanton (48), however, shows close agreement between the moduli of the 2 types of specimens.

The most likely reason for the tendency of E_p to be greater than E_r is that the pulse velocity is less affected by porosity than by the resonance frequency. This is supported by Figure 5 that indicates the loosening of the internal structure of concrete to be caused by repeated freezing and thawing, which reduces the value of E_p to less than that of E_r .

Static Versus Dynamic Moduli

The dynamic Young's moduli are, as a rule, higher than the corresponding static values. Powers was the first to publish comparisons between the static and resonance moduli of concrete (19). He found good agreement between these 2 types of moduli both when the static values were determined by the compression of 6- by 12-in. cylinders and, in another series, when static secant moduli, at about one-third the ultimate strength, were calculated from central deflections of plain concrete beams. (No corrections were applied for shear or stress distribution.) In calculating the resonance modulus, however, he used an oversimplified formula. Had he used the formula corrected by Pickett (20), he would have obtained 7 to 10 percent higher values for E_r . In accordance with this, later investigators find that, by using Pickett's equations, the resonance moduli are regularly higher than the comparable static secant moduli. This difference is influenced by numerous factors, such as the composition and age of concrete and the curing and testing conditions.

Stanton (48) reports that the secant modulus of 6- by 12-in. cylinders at 1,000 psi was found to range from 62 percent of the resonance modulus at 28 days to approximately 75 percent at later periods. Data by Shideler (41) show that values of the resonance

moduli of his moist-cured structural lightweight concretes are about 350,000 psi greater than the secant moduli, whereas this difference for sand-and-gravel concretes is about 1,500,000 psi. He also demonstrates that, as the concretes dry, the difference between the secant modulus and resonance modulus is reduced. Similarly, Reichard (49) finds that the average ratios of the resonance to the secant modulus at the ages of 1 day, 28 days, and 1 year are 1.13, 1.11, and 1.10 respectively for lightweight concretes and 1.37, 1.32, and 1.16 respectively for normal-weight concretes. Data by Hirsch (50) demonstrate that the secant moduli are approximately 10 percent less than the resonance moduli. Chevdeville (51) suggests that the E_{st}/E_r ratio increases to unity as the stress related to the E_{st} decreases. Klieger's pertinent comparison (15) shows that for 6- by 12-in. cylinders made of a wide range of concrete composition, the static moduli are, in all cases, lower than the related resonance moduli, the differences being smaller at 1 and 3 years than at 28 days. Other experimental evidence also indicates (52, 53, 54) that the higher the modulus of elasticity is the closer the agreement between resonance and static moduli will be.

A few comparisons are also available between values of static secant modulus computed by Eq. 2 or 3 from deflection measurements to the related resonance modulus. Philleo noticed a tendency for such E_r/E_{st} ratios to decrease as the modulus of elasticity increased (28). Klieger reports (15) that such static moduli are slightly lower at 28 days than the resonance moduli, but at 1 and 3 years these differences are small and irregular. Thus, it seems that on the average the static secant moduli from deflection formulas, corrected for shear, agree fairly well with the resonance moduli.

Some experimental evidence seems to indicate that the ratio of the pulse modulus to the static modulus is close to the unity when the static modulus is about 6×10^6 psi and increases rapidly with the decrease of the static modulus (28). A group of aerated concrete in the range of 450 to 1,000 psi compressive strength also provided practically the unity for the E_p/E_{st} ratio (55).

There is nothing surprising in the fact that the static and dynamic moduli of elasticity, as a rule, differ. In the static test, there is always an inelastic portion of the measured strain at a particular stress that reduces the calculated value of the modulus. The magnitude of this inelastic portion varies with age and composition of concrete, curing, testing conditions, and other variables. In the dynamic tests, the fundamental resonance frequency or the pulse velocity is determined at a very low stress level of 1 psi. Consequently, the inelastic deformations are almost completely eliminated. For this reason, several investigators consider the dynamic modulus as equivalent to the static initial tangent modulus. Also, the static modulus and the dynamic modulus are affected by the composite character of concrete in different ways. Experimental evidence seems to indicate, for instance, that the value of Young's modulus determined statically is more dependent on the elastic properties of the paste and less on the properties of aggregate than the modulus determined dynamically (28).

Poisson's Ratios

Several investigators have reached the conclusion that dynamic Poisson's ratios are consistently higher than the corresponding static values. Also, there seems to be a difference between dynamic Poisson's ratios obtained by different methods.

Anson and Newman (31) are of the opinion that dynamic Poisson's ratios calculated by Eq. 1 from E_r and G_r flexural and torsional resonance moduli may be 0.03 to 0.05 higher than the corresponding dynamic ratios calculated from pulse velocity and longitudinal resonance frequency tests. Simmons (57) as well as Elvery (53) report that ratios calculated by Eq. 1 from longitudinal and torsional resonance frequencies are consistently about 0.04 lower than the corresponding values from pulse velocity and longitudinal resonance frequency measurements, which, in turn, are about 0.08 higher than the corresponding static ratios. Data by Shideler (41) show that the dynamic Poisson's ratio calculated again by Eq. 1 from flexural and torsional resonance frequencies is, on the average, 0.02 higher for structural lightweight concretes, and 0.03 higher for sand-and-gravel concretes than the static ratio in the case of wet concretes. In the case of dry concretes this difference is practically negligible.

The source of the differences between various Poisson's ratios is similar to that for the various Young's moduli. A significant portion of the difference between static and dynamic ratios is probably due to the inelastic deformations that occur when loads are applied in the static test. Thus, the dynamic Poisson's ratio might be considered as equivalent to the tangent static ratio at zero stress. Also, the static Poisson's ratio and the dynamic ratio are affected by the composite character of concrete in different ways. The dynamic ratio appears to decrease with an increase in aggregate content and age of concrete but increases as the water-cement ratio is increased, whereas the static ratio is hardly affected by the water-cement ratio and increases toward the dynamic value with the age of concrete (56).

A possible reason for the differences between various dynamic Poisson's ratios is that the application of Eq. 1, as well as Eqs. 4 through 11, to concrete is an oversimplification because concrete is not an elastic, homogeneous, and isotropic material.

REPRODUCIBILITY OF THE MEASUREMENTS OF ELASTIC CONSTANTS

The few available experimental data appear to indicate that the reproducibility of the static elastic constants is poorer than that of the dynamic constants. In a comparative test series Mather (58) finds that pulse velocity determinations in uncracked concrete could generally be reproduced within 2 percent by various instruments and operators for mass structures. Wider variation may be expected for short path lengths of the order of 1 ft or less. Witte and Price (42) report average coefficients of variation of 2.3, 7.4, and 10.3 percent for the resonance Young's modulus, compressive secant, and flexural secant moduli respectively.

Simmons (57) shows that the standard deviations of static Poisson's ratio measurements, of dynamic ratios calculated by Eq. 1, and of dynamic ratios calculated from pulse velocity and resonance frequency measurements are 0.014, 0.007, and 0.009 respectively. These values correspond to coefficients of variation of 8.9, 3.6, and 3.8 percent respectively. McCoy and Mather (59) report that the coefficients of variation of dynamic Poisson's ratios calculated by Eq. 1 are 11 percent at the age of 14 days and 9 percent at 180 days. Their specimens all contained the same aggregate, but the types of the applied cements varied within wide limits.

The variation in the elastic constants of concrete in situ can be, of course, much higher. Neville (60) reports cases after Elvery where the ratio of maximum to minimum value of the modulus of elasticity is as high as two.

CONCLUSIONS

1. The shape of the stress-strain diagram for concrete can be derived only qualitatively from the available theories. Empirical formulas can be used for numerical approximation.
2. There are several methods for the determination of each of the elastic constants, and the results are more or less different. As a rule, the constants determined by dynamic methods are greater than the corresponding static constants.
3. A part of these differences is due to the fact that the stresses induced in the dynamic testing are much smaller than those in the static testing. Another reason is that the concrete is not an elastic, homogeneous, or isotropic material.
4. Static tests for the elastic constants involve the application of stresses of the same order as those in practice. Thus, the static constants are of greater significance to the design engineer than the corresponding dynamic values. On the other hand, the determination of the dynamic values is easier and more precise.
5. The reproducibility of the dynamic elastic constants is better than that of the static constants.

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