A Simple Off-Ramp Traffic Model

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This paper gives the analysis of traffic dynamics by formulating a simple traffic model, wherein the presence of an off-ramp is studied on both the upstream and the downstream sides of the exit point. These studies provide a first look at the freeway traffic dynamics in the vicinity of an off-ramp. The analytical formulation is based solely on the primary governing mechanisms that are thought to contribute significantly to the traffic behavior in the vicinity of an off-ramp. The present analysis has incorporated a simple weaving effect on the upstream side and a simple right-lane back merge on the downstream side of a unidirectional 2-lane freeway. The analysis technique presented in this paper can be easily modified for application to other multilane freeways.

•THE PRESENCE OF AN OFF-RAMP on a unidirectional freeway generally induces some complex lane-changing effects in the traffic on the upstream side of the exit point. Those drivers in the left lanes who want to leave the freeway at a certain off-ramp start merging into the right lane. This type of lane-changing of vehicles from left to right lane and vice versa is also known as the weaving effect and is a typical phenomenon of traffic dynamics on the upstream side of the off-ramp. Immediately after the exit on the downstream side of the off-ramp there is a region of very low traffic density in the right lane. This is caused by vehicles having left this lane to use the exit ramp. This low density in the right lane is then stabilized as some cars from the left lanes move toward the right lane, finally resulting in traffic equilibrium in all lanes.

The analysis of an off-ramp traffic model described in this paper is complementary to the previous analysis of on-ramp models (1, 2, 3) and is based on the expected-value continuum principles. This paper is written to investigate off-ramp traffic dynamics, and the analytical formulation is based solely on the primary governing mechanisms thought to contribute significantly to traffic behavior in the vicinity of an off-ramp. It is hoped that theoretical considerations of the present discussion may assist in making pertinent experimental observations to determine the most desirable traffic features for consideration in a more elaborate theory describing off-ramp traffic dynamics.

FORMULATION OF AN OFF-RAMP TRAFFIC MODEL

In this study of the traffic dynamics in the upstream and downstream sides of the off-ramp, the analysis is based on an idealized 2-lane freeway with no other points of exit or entrance except the off-ramp under consideration. It is assumed that the traffic is under equilibrium conditions at some great distances on the upstream and downstream sides of the off-ramp under consideration, such that $k_{i0}^{U}(t)$ = equilibrium density in lane i that exists at some great distance x in the upstream side of the off-ramp at time t, and $k_{i0}^{D}(t)$ = equilibrium density in lane i that exists at some great distance x in the downstream side of the off-ramp at time t. Lanes 1 and 2 are the right and left lanes respectively.

It is further assumed that α percentage of the cars under equilibrium in lane 1 and β percentage of cars under equilibrium in lane 2 will be leaving the freeway through

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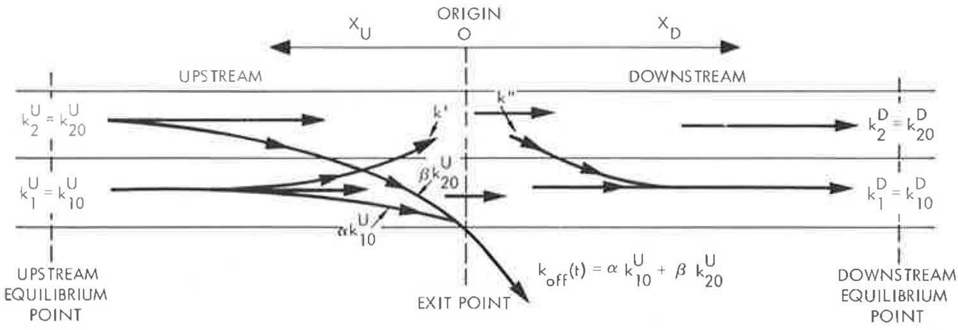

Figure 1. Schematic diagram of off-ramp traffic dynamics on the upstream and downstream sides.

the exit ramp under consideration. Because the flow at some great distance on the upstream side of the off-ramp is assumed to be constant, we have the following relationship between the equilibrium densities that prevail at great distances on the upstream and downstream sides of the off-ramp:

$$k_{10}^U + k_{20}^U = k_{10}^D + k_{20}^D + \alpha k_{10}^U + \beta k_{20}^U \tag{1}$$

Figure 1 shows the schematic configuration of traffic movements on the upstream and downstream sides of the off-ramp. It may be noted from this figure that we take the off-ramp as the origin point for the distance coordinate. On the upstream side the distance increases in the opposite direction of the traffic flow and on the downstream side it increases in the direction of traffic flow.

In setting up the appropriate boundary conditions, we assumed that all the vehicles from the left lane that are leaving the freeway have moved into the right lane at the exit point of the off-ramp, so that

$$\alpha k_{10}^U(t) + \beta k_{20}^U(t) = K(0, t) = K_{off}(t) \tag{2}$$

and $K_{off}(t)$ = vehicle density leaving the freeway.

In order to counterbalance some effect of the movement of vehicles from left to right lane, we assumed that some vehicles move gradually from right to left lane. This movement of vehicles is taken in accordance with the movements of vehicles from left to right lane, and its value is assumed to reach $k'(t)$ at the exit point of the off-ramp.

On the downstream side of the off-ramp, the vehicles from the left lane move back to the right lane to occupy the empty space created by the vehicles that left the freeway. The density of vehicles $k''(t)$ that move from left to right lane on the downstream side of the off-ramp would be close to $k'(t)$ if

$$\alpha/\beta = 1 \tag{3}$$

However, if α/β is larger than 1, a relatively larger proportion of drivers would be traveling in the right lane at the upstream equilibrium point of the off-ramp. Consequently the vehicle density $k''(t)$ that moves from the left to the right lane on the downstream side of the off-ramp is larger than the vehicle density $k'(t)$ that has moved from the right to the left lane on the upstream side of the off-ramp. The subsequent analysis in this paper is based on the equation of continuity of flow and postulates that the transfer of cars between lanes is proportional to the density difference between 2 adjacent lanes. This type of lane-changing behavior has been termed the uniform lane-changing hypothesis (1). The basic set of expected-value equations of continuity for the 2-lane freeway has been derived in the Appendix and can be written as

$$\left.\begin{aligned} D_tK_1 + cD_xK_1 &= a(K_2 - K_1) \\ D_tK_2 + cD_xK_2 &= a(K_1 - K_2) \end{aligned}\right\} \tag{4}$$

where

D_t, D_x = partial derivative operators $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$ with regard to t and x respectively;
K_i = density perturbation in lane i;
c = wave speed; and
a = positive-valued constant of dimension time^{-1}.

This set of equations are applicable to uniform freeways, where the $\bar{v}_i$ - k_i (speed-density) relationship for equilibrium conditions is the same for each lane and equilibrium density is the same across lanes. These equations are also applicable to non-uniform freeways where only the density perturbations are treated to fluctuate by a uniform lane-changing hypothesis together with the same wave speeds in both lanes. The density perturbation in lane i is related to the equilibrium density by

$$K_i(x, t) = k_i(x, t) - k_{i0} \tag{5}$$

where

$k_i(x, t)$ = traffic density in lane i, and
k_{i0} = equilibrium density in lane i.

It can be seen that integration of Eq. 5 over a given road section gives

$$N_i(t) = n_i(t) - n_{i0} \tag{6}$$

where

$n_i(t)$ = total number of vehicles in lane i;
n_{i0} = total number of vehicles in lane i at equilibrium; and
$N_i(t)$ = perturbation number of vehicles in lane i about equilibrium.

THE GENERAL PERTURBATION MATRIX MODULE

Before the effect of the traffic movements on the upstream and downstream sides of the off-ramp is examined, it is necessary to develop—with the help of Eq. 4—a general traffic density perturbation matrix module. With the help of such a module, the various specific traffic movements can be studied in the vicinity of the off-ramp defining the various associated boundary conditions. The density perturbation matrix

$$K(x, t) = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{7}$$

and the matrix of coefficients of density perturbations

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8}$$

are introduced, and Eq. 4 can be written in the following compact form:

$$(D_t + cD_x)K + aAK = 0 \tag{9}$$

If M is a modal matrix of A and M^{-1} is the inverse of M, we have the property

$$M^{-1}AM = S \tag{10}$$

where X is diagonal spectral matrix of A and has the form

$$S = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{11}$$

where λ_1, λ_2 are the eigenvalues of A. A simple calculation shows (2) that the modal matrix M of A is

$$M = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{12}$$

and its inverse M^{-1} is

$$M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{13}$$

The product of $M^{-1}AM$ then gives the following spectral matrix S:

$$M^{-1}AM = S = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \tag{14}$$

The eigenvalues of A are therefore $\lambda_1 = 0$ and $\lambda_2 = 2$.

In order to get a general perturbation matrix module, we need to solve Eq. 9. The solution of Eq. 9 is facilitated by introducing the following linear transformation:

$$K = My \tag{15}$$

where y is some new column matrix.

If Eq. 15 is substituted into Eq. 9, the result is

$$(D_t + cD_x)My + aAMy = 0 \tag{16}$$

We now premultiply Eq. 16 by M^{-1} to obtain

$$(D_t + cD_x)y + aSy = 0 \tag{17}$$

Because S is the diagonal matrix of A, the elements of the column y are uncoupled, and we have

$$\left.\begin{aligned} (D_t + cD_x)y_1 &= 0 \\ (D_t + cD_x)y_2 + 2ay_2 &= 0 \end{aligned}\right\} \tag{18}$$

The solutions of Eq. 18 take the form

$$\left.\begin{aligned} y_1(x, t) &= y_{10}(t - \frac{x}{c}) \\ y_2(x, t) &= e^{-(2ax)/c} y_{20}(t - \frac{x}{c}) \end{aligned}\right\} \tag{19}$$

where

$y_{10}(t) = y_1(0, t)$, and

$y_{20}(t) = y_2(0, t)$ (20)

By introducing the matrices

$$Y_0(t) = \begin{bmatrix} y_{10}(t) \\ y_{20}(t) \end{bmatrix} \tag{21}$$

and

$$K_0(t) = \begin{bmatrix} K_1(0, t) \\ K_2(0, t) \end{bmatrix} = \begin{bmatrix} K_{10}(t) \\ K_{20}(t) \end{bmatrix} \tag{22}$$

we get the relation

$$y_0(t) = M^{-1}K_0(t) \tag{23}$$

The general perturbation matrix module can then be written as

$$K(x, t) = MB(x)M^{-1}K_0\left(t - \frac{x}{c}\right) \tag{24}$$

where

$$B(x) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(2ax)/c} \end{bmatrix} \tag{25}$$

and $K_0(t)$ is the initial value perturbation matrix.

TRAFFIC DYNAMICS ON THE UPSTREAM SIDE

Figure 1 shows a schematic representation of traffic movements on the upstream and downstream sides of the off-ramp. It can be seen from this figure that the distance coordinate has its origin at the exit point of the off-ramp and increases against the direction of flow on the upstream side. It is assumed that the flow is under equilibrium conditions with normal traffic dynamics at some great distance on the upstream side of the off-ramp. As mentioned earlier, an expected α percentage of cars under equilibrium in the right lane and an expected β percentage of cars under equilibrium in the left lane would be leaving the freeway through the off-ramp. We shall now formulate the movements of β percentage of the cars under equilibrium in the left lane to merge into the right lane so that they leave the freeway at the exit point of the off-ramp. This can be accomplished by incorporating the previously mentioned boundary condition in the general perturbation matrix module given in Eq. 24. The initial conditions for the density perturbations will then be given as

$$K_0(t) = \begin{bmatrix} K_{10}(t) \\ K_{20}(t) \end{bmatrix} = \begin{bmatrix} -\alpha k_{10}^{U}(t) \\ -\beta k_{20}^{U}(t) \end{bmatrix} \tag{26}$$

Corresponding to the initial conditions, the matrix density perturbation module takes the following values:

$$K^{I}(x, t) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-(2ax)/c} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\alpha k_{10}^{U}\left(t + \frac{x}{c}\right) \\ -\beta k_{20}^{U}\left(t + \frac{x}{c}\right) \end{bmatrix} \tag{27}$$

The argument in the initial value perturbation matrix is $(t + \frac{x}{c})$, because the distance coordinate increases in the opposite direction of the flow.

Simplifying Eq. 27 gives

$$K^{I}(x, t) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} e^{-(2ax/c)} \\ 1 & -\frac{1}{\sqrt{2}} e^{-(2ax/c)} \end{bmatrix} \begin{bmatrix} -\frac{\alpha}{2} k_{10}^{U}\left(t + \frac{x}{c}\right) - \frac{\beta}{2} k_{20}^{U}\left(t + \frac{x}{c}\right) \\ -\frac{\alpha}{\sqrt{2}} k_{10}^{U}\left(t + \frac{x}{c}\right) + \frac{\beta}{\sqrt{2}} k_{20}^{U}\left(t + \frac{x}{c}\right) \end{bmatrix} \tag{28}$$

In order to counterbalance this surge of traffic flow in the right lane, we shall now incorporate an opposite effect whereby some drivers from the right lane who do not intend to use the off-ramp will move into the left lane. This type of analysis brings about the weaving effect, which is a well-known fact in the upstream side of the off-ramp. It has been assumed that an estimated $k'(t)$ vehicle density takes part in such movements and the initial conditions for the density perturbation bringing about this effect are given by

$$K_0(t) = \begin{bmatrix} -k'(t) \\ k'(t) \end{bmatrix} \tag{29}$$

A simple analysis gives the following value to the matrix perturbation module of the type given in Eq. 28:

$$K^{II}(x, t) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} e^{-(2ax/c)} \\ 1 & -\frac{1}{\sqrt{2}} e^{-(2ax/c)} \end{bmatrix} \begin{bmatrix} 0 \\ -\sqrt{2}\, k'\left(t + \frac{x}{c}\right) \end{bmatrix} \tag{30}$$

Superposition of Eqs. 28 and 30 gives the following weaving perturbation density expression, which describes the traffic movements on the upstream side of the off-ramp:

$$K^{U}(x, t) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} e^{-(2ax/c)} \\ 1 & -\frac{1}{\sqrt{2}} e^{-(2ax/c)} \end{bmatrix} \left\{ \begin{bmatrix} -\frac{\alpha}{2} k_{10}^{U}\left(t + \frac{x}{c}\right) - \frac{\beta}{2} k_{20}^{U}\left(t + \frac{x}{c}\right) \\ -\frac{\alpha}{\sqrt{2}} k_{10}^{U}\left(t + \frac{x}{c}\right) + \frac{\beta}{\sqrt{2}} k_{20}^{U}\left(t + \frac{x}{c}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ -\sqrt{2}\, k'\left(t + \frac{x}{c}\right) \end{bmatrix} \right\} \tag{31}$$

Equation 31 with the incorporation of the relation given in Eq. 5 gives the following expression for each lane that describes the traffic movements on the upstream side of the off-ramp:

$$\left.\begin{aligned} k_1^U(x_U, t) &= k_{10}^U(x_U, t) - \frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) \\ &\quad - \left[\frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) + k'\left(t + \frac{x_U}{c}\right)\right] e^{-(2ax_U/c)} \\ k_2^U(x_U, t) &= k_{20}^U(x_U, t) - \frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) \\ &\quad + \left[\frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) + k'\left(t + \frac{x_U}{c}\right)\right] e^{-(2ax_U/c)} \end{aligned}\right\} \tag{32}$$

where x_U denotes the distance measured on the upstream side as shown in Figure 1.

TRAFFIC DYNAMICS ON THE DOWNSTREAM SIDE

The traffic dynamics on the downstream side of the off-ramp are relatively simple as compared to the upstream side. There is no weaving effect on the downstream side of the off-ramp: Here some of the drivers in the left lane simply move back to the right lane and occupy the empty space created by the vehicles that left the freeway via the right lane. At some great distance on the downstream side of the off-ramp the vehicles reach equilibrium conditions, and here again the normal traffic dynamics are established in each lane. The equilibrium densities in the two lanes are k_{10}^D and k_{20}^D at that location of the freeway. As mentioned earlier, the expected vehicle density that moves from left to right lane is assumed to be $k''(t)$. Such boundary conditions can be easily incorporated by making use of the general perturbation matrix module given in Eq. 24. The initial conditions for the density perturbation will then be given as

$$K_0(t) = \begin{bmatrix} K_{10}(t) \\ K_{20}(t) \end{bmatrix} = \begin{bmatrix} k''(t) \\ -k''(t) \end{bmatrix} \tag{33}$$

Corresponding to the initial conditions, the matrix perturbation given in Eq. 24 takes the following value:

$$K(x, t) = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ e^{-(2ax)/c} & \frac{e^{-(2ax)/c}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ -\sqrt{2}\,k''\left(t - \frac{x}{c}\right) \end{bmatrix} \tag{34}$$

Equation 34 with the incorporation of the relation given in Eq. 5 gives the following expressions for each lane that describe the traffic movements on the downstream side of the off-ramp:

$$\left.\begin{aligned} k_1^D(x_D, t) &= k_{10}^D - k''\left(t - \frac{x_D}{c}\right)e^{-(2ax_D)/c} \\ k_2^D(x_D, t) &= k_{20}^D + k''\left(t - \frac{x_D}{c}\right)e^{-(2ax_D)/c} \end{aligned}\right\} \tag{35}$$

where x_D denotes the distance measured on the downstream side of the off-ramp as shown in Figure 1. Also, as mentioned earlier, if the number of drivers in each lane who are leaving the freeway through the exit ramp are proportionate to the equilibrium densities that exist at some great distance on the upstream side, then the expected vehicle density that moves from left to right lane in the downstream side of the off-ramp would be very close to k′(t). Corresponding to this situation, Eq. 35 will take the following form:

$$\left.\begin{aligned} k_1^D(x_D, t) &= k_{10}^D - k'\left(t - \frac{x_D}{c}\right)e^{-(2ax_D)/c} \\ k_2^D(x_D, t) &= k_{20}^D + k'\left(t - \frac{x_D}{c}\right)e^{-(2ax_D)/c} \end{aligned}\right\} \tag{36}$$

The corresponding expression for density distribution on the upstream side can also be written in terms of the equilibrium densities k_{10}^D, k_{20}^D that prevail at some great distance on the downstream side of the off-ramp. This is achieved by making use of the relationship between the equilibrium densities on the upstream and downstream sides of the off-ramp with the expected number of vehicles leaving the freeway as follows:

$$\left.\begin{aligned} k_{10}^U &= \alpha k_{10}^U + k_{10}^D \\ k_{20}^U &= \beta k_{20}^U + k_{20}^D \end{aligned}\right\} \tag{37}$$

Substitution of Eq. 37 in Eq. 32 gives

$$\left.\begin{aligned} k_1^U &= k_{10}^D + \alpha k_{10}^U - \frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) \\ &\quad - \left[\frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) + k'\left(t + \frac{x_U}{c}\right)\right]e^{-(2ax_U/c)} \\ k_2^U &= k_{20}^D + \beta k_{20}^U - \frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) \\ &\quad + \left[\frac{\alpha}{2}k_{10}^U\left(t + \frac{x_U}{c}\right) - \frac{\beta}{2}k_{20}^U\left(t + \frac{x_U}{c}\right) + k'\left(t + \frac{x_U}{c}\right)\right]e^{-(2ax_U/c)} \end{aligned}\right\} \tag{38}$$

Equations 38 and 36 (or 35) decribe the traffic movements on the upstream and downstream sides of the off-ramp. It may be noted that these 2 equations observe the regional boundary conditions and are identical at the exit point of the off-ramp, when both x_U and x_D tend to zero.

CONCLUSIONS

The traffic dynamics on a uniform 2-lane freeway due to the presence of an off-ramp have been given for both the upstream and downstream sides of an off-ramp. The present analysis is developed on the basis of a simple expected-value continuum model and assumes that the rate of exchange of vehicles between the 2 lanes is proportional to the difference in their densities and that the present form of the equation of continuity applies. Also, the analysis assumes that the expected density of vehicles leaving the freeway is small compared with the density on the freeway. The results of the present analysis show that the freeway vehicle dynamics involve more lane-changing maneuvers near the vicinity of an off-ramp and are exponentially adjusted to normal lane-changing maneuvers on both upstream and downstream sides of the off-ramp. The method of analysis presented in this paper can be easily modified to take into account other multilane freeways. The mathematical analysis is simple and flexible so that a more elaborate traffic feature can easily be incorporated. However, before we attempt to incorporate other complexities, experimental data are necessary to validate the present model and to bring out other significant traffic features that may be desirable for developing a more realistic and elaborate off-ramp traffic model.

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Appendix

DERIVATION OF BASIC SET OF EXPECTED-VALUE EQUATIONS OF CONTINUITY

The basic set of expected-value equations of continuity for traffic flow can be derived on the concept of speed distribution function. The speed distribution function $f(x, v, t)$, for a given concentration, has been defined such that $f(x, v, t)dxdv$ = the expected number of vehicles at time t whose actual speeds are between v and $v + dv$ in the road interval between x and $x + dx$. Thus,

$$\int_0^\infty f(x, v, t)dv = k(x, t) \tag{39}$$

where $k(x, t)$ is the concentration at point x and time t. The average speed $\overline{v}(x, t)$ and flow $q(x, t)$ are then

$$\overline{v}(x, t) = \frac{\int_0^\infty vf(x, v, t)dv}{\int_0^\infty f(x, v, t)dv} \tag{40}$$

$$q(x, t) = \int_0^\infty vf(x, v, t)dv \tag{41}$$

and

$$\int_{x_1}^{x_2} \int_0^\infty f(x, v, t)dvdx = N_{1,2}(t) \tag{42}$$

where $N_{1,2}$ is the total number of vehicles at time t between positions x_1 and x_2 on the road.

Figure 2 shows a small area element, dxdv, that represents a velocity interval dv and a road piece of length dx. With the help of this figure, we wish to evaluate the time rate of change of the speed distribution function $f(x, v, t)$ by using the vehicle conservation principle. The combination of this time rate of change of the speed distribution function and the relationships given in Eqs. 39, 40, and 41 would give us the expected-value equation of continuity for traffic flow.

The horizontal arrows shown in Figure 2 represent the movement of traffic flow, and the difference in the number of cars at the inflow boundary x and outflow boundary

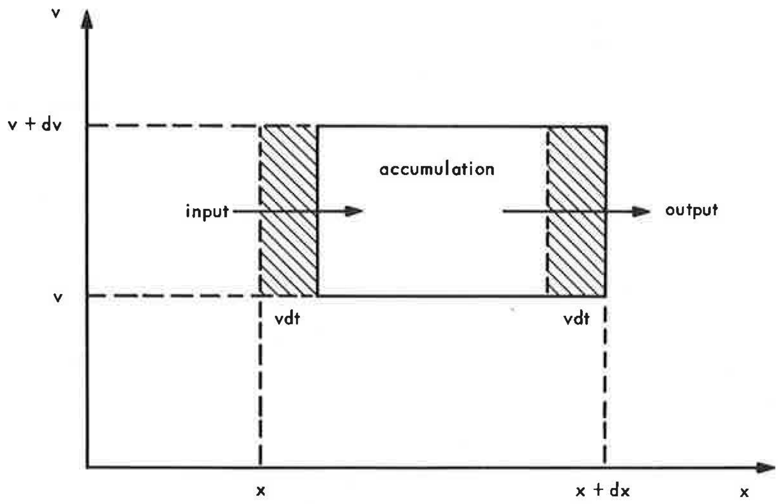

Figure 2. Segment of a road and a speed band to illustrate the derivation of an expected-value equation of continuity for traffic flow.

x + dx points occurs because of the change in the speed distribution function f(x, v, t). Hence, expected influx of vehicles at x in the time interval dt = +vdtdvf(x, v, t); expected efflux of vehicles at x + dx in time interval dt = -vdtdvf(x + dx, v, t); and net accumulation of net change in time dt of the expected number of vehicles in dx, dv-element = $\frac{\partial f(x, v, t)}{\partial t}$ dxdvdt. With the vehicle conservation principle, input - output = accumulation,

$$v\,dt\,dv\,f(x, v, t) - v\,dt\,dv\,f(x + dx, v, t) = \frac{\partial f(x, v, t)}{\partial t}\,dx\,dv\,dt$$

Simplifying, we have

$$-v\frac{f(x + dx, v, t) - f(x, v, t)}{dx} = \frac{\partial f(x, v, t)}{\partial t}$$

that, on taking the limit as $\Delta x \to 0$, becomes

$$\frac{\partial f(x, v, t)}{\partial t} + v\frac{\partial f(x, v, t)}{\partial x} = 0 \qquad (43)$$

Integration of Eq. 43 over all speeds yields

$$\int_0^{\infty}\left[\frac{\partial f(x, v, t)}{\partial t} + v\frac{\partial f(x, v, t)}{\partial x}\right]dv = 0$$

or

$$\frac{\partial}{\partial t}\int_0^{\infty} f\,dv + \frac{\partial}{\partial x}\int_0^{\infty} v f\,dv = 0$$

Using Eqs. 39 and 41 gives

$$\frac{\partial k(x, t)}{\partial t} + \frac{\partial q(x, t)}{\partial x} = 0$$

or

$$D_t k + D_x q = 0 \tag{44}$$

where D_t and D_x are the partial derivatives with regard to t and x respectively. Equation 44 is the expected-value equation of continuity for traffic flow.

Now let Q(x, t) denote the net flow per mile entering or leaving the highway. It takes the positive value when the flow is entering and negative when the flow is leaving. The corresponding equation of continuity can be written as

$$D_t k + D_x q = Q(x, t) \tag{45}$$

When the highway consists of 2 unidirectional lanes, then it is possible for an interchange of vehicles to occur between the 2 lanes, and we may write 2 equations of continuity, one for each lane, of the form

$$D_x q_1 + D_t k_1 = Q_1 \tag{46}$$

$$D_x q_2 + D_t k_2 = Q_2 \tag{47}$$

where

Q_1 = net flow per unit distance of vehicles entering or leaving lane 1, and
Q_2 = net flow per unit distance of vehicles entering or leaving lane 2,

such that

$$k_i(x, t) = \int_0^{\infty} f_i(x, v, t)\,dv \tag{48}$$

$$q_i(x, t) = \int_0^{\infty} v f_i(x, v, t)\,dv \tag{49}$$

where i denotes the lane number. If there are no on-ramps and off-ramps, so that no vehicles are leaving or entering the road, we must have

$$Q_2 = -Q_1 \tag{50}$$

In order to estimate the equilibrium density k_{i0} in the i th lane in terms of the equilibrium (or average steady-state) speeds $\overline{v}_{i0}$, we shall assume that each lane satisfies the following generalized speed density relation so that we have

$$\overline{v}_{i0} = \overline{v}_i^0 \left(1 - \frac{k_{i0}}{k_j}\right)^n \tag{51}$$

where

$\overline{v}_i^0$ = desired average speed for lane i;
k_j = jam traffic concentration; and
n = some real number whose magnitude is greater than zero.

Equation 51 may be solved for k_{i0} to obtain

$$k_{i0} = k_j \left[1 - \left(\frac{\bar{v}_{i0}}{\bar{v}_i^0} \right)^{1/n} \right] \tag{52}$$

If $f_{i0}(v)$ and $f_i^0(v)$ are the actual and desired speed distribution functions for the i th lane at equilibrium conditions, we have

$$\bar{v}_{i0} = \frac{\int_0^\infty f_{i0}(v) v dv}{\int_0^\infty f_{i0}(v)\, dv}, \quad \bar{v}_i^0 = \frac{\int_0^\infty f_i^0(v) v dv}{\int_0^\infty f_i^0(v) dv} \tag{53}$$

$$k_{i0} = \int_0^\infty f_{i0}(v) dv = \int_0^\infty f_i^0(v) dv \tag{54}$$

where all the speed distributions are realized at steady state. It will be assumed now that the instantaneous speed distribution function $f_i(x, v, t)$ for the i th lane is given by

$$f_i(x, v, t) = f_{i0}(v) + F_i(x, v, t) \tag{55}$$

where

$F_i(x, v, t)$ = perturbation of the speed distribution function in lane i.

Integrating Eq. 55 over all speeds gives

$$k_i(x, t) = k_{i0} + K_i(x, t) \tag{56}$$

We now postulate that $Q_1(x, t)$, the net flow per unit distance of vehicles entering or leaving lane 1, is

$$\begin{aligned} Q_1(x, t) &= a\left\{[k_2(x, t) - k_1(x, t)] - (k_{20} - k_{10})\right\} \\ &= a[K_2(x, t) - K_1(x, t)] \end{aligned} \tag{57}$$

where a is a positive constant with dimension time^{-1}. Equation 57 gives $Q_1(x, t) = 0$ when $k_2(x, t) = k_{20}$ and $k_1(x, t) = k_{10}$ and gives a positive flow when $[k_2(x, t) - k_1(x, t)]$ exceeds $(k_{20} - k_{10})$. This relation for Q_i expresses simply a behavior in which drivers tend to leave a more crowded lane and enter a less crowded one. This type of lane-changing behavior is defined as the uniform lane-changing hypothesis. With Eq. 57 and the car conservation relationship given in Eq. 50, we get the continuity equation for a 2-lane unidirectional highway as

$$\begin{aligned} D_x q_1 + D_t k_1 &= a[(k_2 - k_1) - (k_{20} - k_{10})] \\ &= a(K_2 - K_1) \end{aligned} \tag{58}$$

$$\begin{aligned} D_x q_2 + D_t k_2 &= a[(k_1 - k_2) - (k_{10} - k_{20})] \\ &= a(K_1 - K_2) \end{aligned} \tag{59}$$

In order to study the traffic dynamics, it is necessary to assume that the flow in each lane is a function of the density in that lane and of the position x along the highway; namely

$$q_i = q_i(k_i, x) \quad i = 1, 2 \tag{60}$$

If the road is space homogeneous, the flow will not depend on the position, and Eq. 60 reduces to

$$q_i = G_i(k_i) \tag{61}$$

Equation 61 is regarded as the "equation of state" of the traffic fluid. Partial differentiation with respect to x of Eq. 61 gives

$$D_x q_i = \left(\frac{dG_i}{dk_i}\right) D_x k_i \quad i = 1, 2 \tag{62}$$

Introducing the notation

$$c_1 = \frac{dG_1}{dk_1} \qquad c_2 = \frac{dG_2}{dk_2} \tag{63}$$

we get

$$D_x q_i = c_i D_x k_i \quad i = 1, 2 \tag{64}$$

c_1 and c_2 are the velocities of wave propagation.

If we now substitute the relationship of Eqs. 56 and 64 into Eqs. 58 and 59, we get the equations

$$D_t K_1 + c_1 D_x K_1 = a(K_2 - K_1) \tag{65}$$

$$D_t K_2 + c_2 D_x K_2 = a(K_1 - K_2) \tag{66}$$

If the density perturbations are small, we can take the values of wave speeds at equilibrium conditions and assume them to hold for the traffic densities under consideration, such that

$$c_1 = \left(\frac{dq_1}{dk_1}\right)_{k=k_{10}} \qquad c_2 = \left(\frac{dq_2}{dk_2}\right)_{k=k_{20}} \tag{67}$$

For a uniform freeway, where the $\overline{v}_i - k_i$ relationship for equilibrium conditions is the same for each lane and equilibrium density is the same across lanes, we get the following equilibrium equations:

$$\overline{v}_1 = \overline{v}_1^0 \left[1 - \left(\frac{k_{10}}{k_j}\right)\right]^n = \overline{v}_2 \tag{68}$$

$$k_{10} = k_j \left[1 - \left(\frac{\overline{v}_1}{\overline{v}_i^0}\right)^{1/n}\right] = k_{20} \tag{69}$$

$$q_1 = \overline{v}_1^0 \left(k_{10}^{1/n} - \frac{k_{10}^{(n+1)/n}}{k_1} \right)^n = c_2 \tag{70}$$

The wave velocities c_1 and c_2 given by Eq. 67 would have the values

$$c_1 = \overline{v}_1^0 \left(1 - \frac{k_{10}}{k_j}\right)^{n-1} \left(1 - (n+1)\frac{k_{10}}{k_j}\right) = c_2 \tag{71}$$

In this uniform case, Eqs. 65 and 66 take the form (because $c_1 = c_2$)

$$D_t K_1 + c D_x K_1 = a(K_2 - K_1) \tag{72}$$

$$D_t K_2 + c D_x K_2 = a(K_1 - K_2) \tag{73}$$

It may be mentioned here that this set of equations also corresponds to nonuniform freeways, where only the density perturbations are treated to fluctuate by a uniform lane-changing hypothesis along with the same wave speeds in both lanes.