SOLUTIONS AND SOLUTION TECHNIQUES FOR BOUNDARY VALUE PROBLEMS

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One of the practical objectives of the theories of mechanics is to assist in the solution of engineering design problems by providing the theoretical basis for determining the response of a system to a variety of prescribed inputs. This is done through the formulation and solution of boundary value problems. There is one major factor that has, during the last decade, changed the entire approach to the solution of boundary value problems. This is the development of numerical techniques that, in conjunction with the availability of high-speed digital computers, permit the solution of complex boundary value problems. The availability of operational computer programs has made it possible for the average practicing pavement engineer to conduct analyses that only a few years ago would have been considered impractical. One of the major thrusts in achieving progress in pavement design is the use of this capability.

In writing this paper, I find there is the dilemma of whether to strive for completeness in the theoretical aspects or to try to answer in broad terms the two questions of what is the present state of the art in this area and of where the future effort should be. This paper takes the broad approach. It is felt that those whose interest is in the detailed theoretical aspects are sufficiently familiar with recent developments and that a relatively brief summary is not likely to provide them with new information. Furthermore, inclusion of a detailed theoretical discussion is likely to make the paper less readable to those whose major interest is application. The paper attempts to point out those solutions and solution techniques available at the present time in a form that can be used in design. Because one of the major objectives of the workshop is to look toward the future and encourage an exchange of ideas with regard to future research and research in progress, this paper also discusses directions of future research and development. However, before proceeding to a discussion of various methods of solution, I should comment briefly on the formulation of a boundary value problem.

FORMULATION OF THE PROBLEM

The formulation of a boundary value problem involves idealizing the real physical problem and casting it into mathematical form. For the class of problems representative of pavement systems, the mathematical form of the boundary value problem is a set of partial differential equations subject to various initial and boundary conditions.

There are three essential components to a boundary value problem: (a) governing equations, (b) constitutive equations, and (c) boundary and initial conditions. For the analysis of pavements, the governing equations are the equations of equilibrium, motion (for dynamic problems), and compatibility. These equations are derived from the basic laws of classical physics and from continuity considerations in the material. Various approximations can be introduced at this level (e.g., small strains to obtain linearity and symmetry of the stress tensor). It should be recognized that the governing equations are independent of any material properties.

Constitutive equations are representations of the properties of the particular materials under consideration and represent idealizations of actual material behavior.

Boundary conditions may consist of prescribed displacements and stresses on various boundaries. (For thermal and hygro stresses it is necessary to define the temperature and moisture contents as functions of space and time.) For static problems this is sufficient; for dynamic problems it is necessary to specify the conditions at some arbitrary time, generally at t = 0, when they are called initial conditions. The governing and constitutive equations can only be solved in general terms; it is boundary and initial conditions that make the general solution specific for the problem under consideration. The boundary and initial conditions also represent various levels of idealization. For example, the actual time variation of load might be approximated by a simple analytic function (e.g., sine), or nonaxisymmetric loads might be approximated by axisymmetric load distributions.

It is appropriate to comment briefly on how these three components are accounted for in modeling a pavement section. The governing equations are of general applicability. The materials composing the various layers have to be represented by appropriate constitutive equations. This aspect is discussed in a separate paper. (See the Appendix for bibliography.) In addition, the loading conditions and the geometry of the problem have to be approximated in order to make the problem amenable to solution.

The loads are dynamic and can be considered to be applied randomly with regard to both space and time. All current methods of pavement analysis and design treat the loads deterministically. The other basic assumption on loading conditions is whether to treat the loads as static or dynamic. All design methods in use at the present time treat the load as static. Analytical solutions for dynamic problems have been developed on the assumption that inertial effects can be neglected. Analyses have shown that for highway traffic this is a reasonable assumption. These solutions are only of interest when the materials in the pavement system are rate-dependent.

In practice, multiple loads are applied to the pavement. For linear problems the fundamental problem is based on the application of the single load; the multiple load situation can be treated by superposition of single load solutions. The shape of the loaded area and the distribution of the load over this area depend primarily on the tire, the inflation pressure, and the characteristics of the surface layer. All current design methods using analytical solutions make the assumption that the load is distributed uniformly and can be considered axisymmetric. If the physical geometry of the problem is axisymmetric, then nonaxisymmetric load distributions can be analyzed for linear problems.

Assumptions regarding the geometry of the total structural problem depend primarily on the location of the loads relative to the edges of the pavement. This leads to the two assumptions that have been used in the structural analysis of pavements: the layered system theory and the thin plate theory. If the loads are sufficiently distant from the edges of the pavement, in that the effects of the boundary on the stresses in pavement can be neglected, then the problem can be treated as a layered system in which each layer is of infinite horizontal extent. When edge effects are important the extent of the pavement layer has to be considered finite in extent, and the thin plate theory is used because it is not possible to handle the general three-dimensional problem. This latter approach has been used almost exclusively for the design of concrete pavements.

The increased availability of computer programs, which utilize various numerical techniques to solve boundary value problems, has resulted in a tendency to decrease the attention paid to the formulation of the problem. The importance of a correctly formulated boundary value problem cannot be too strongly emphasized. No solution technique, irrespective of the degree of sophistication, can provide adequate answers for design if the problem has been formulated incorrectly.

METHODS OF SOLUTION

There are two basic techniques for obtaining solutions to boundary value problems. These are analytical (sometimes referred to as classical) and numerical. Because the final objective in the case of a practical problem is a numerical result, no solution relies entirely on one of these techniques. A numerical solution technique will use a problem formulation that will be directed toward a computational procedure from the outset, whereas an analytical technique will carry the solution as far as possible before resorting to numerical calculations.

Numerical techniques require that various approximations be made in developing the solution to boundary value problems. Because of the increased availability of "ready-made operational" computer programs, there is a tendency to ignore the effects of possible errors from making discrete and rounding off and questions of instabilities and convergence. Whereas the available programs are satisfactory in the solution of most linear elastostatic problems, these effects must be considered in the analysis of nonlinear and dynamic problems. Possible errors and questions of stability and convergence are discussed briefly in terms of the finite difference and finite element techniques.

As pointed out earlier, the formulation of a boundary value problem is in terms of differential equations. In the finite difference approach, the basic principle is that the derivative can be represented in discrete form. The differential equations are then represented by difference equations. A difference equation approximation must satisfy the requirement that as the mesh size goes to zero the differential equation is obtained. Furthermore, as the mesh size decreases the numerical solution should approach the "exact" solution. In a numerical solution there are errors due to discretization that are dependent on the mesh size and errors due to rounding off that occur because of the truncation of numbers in a computer. An important consideration in numerical techniques is stability; this applies specifically to step-by-step procedures. Because there is some error with each step, the computational scheme should be such that the error does not grow too rapidly.

In the finite element technique, the physical problem is approximated by dividing the solid body into a series of elements. This method, like the finite difference method, produces solutions that have discretization and rounding-off errors. A balance must be obtained between the need for accuracy, which requires a large number of elements, and the need for minimizing computer time, which increases with an increase in the number of elements. For dynamic and nonlinear problems, where step-by-step procedures are used, stability considerations are important.

When available solutions and solution techniques are examined, it is possible to subdivide them into a variety of categories. Because of the emphasis of this paper it is appropriate to consider them in the following two categories: (a) solutions and techniques for linear problems, and (b) solutions and techniques for nonlinear problems. Although this division may appear artificial in that techniques for solving nonlinear problems are generally applicable to linear problems, the subdivision is particularly appropriate when past achievements and future goals are considered.

SOLUTIONS AND SOLUTION TECHNIQUES FOR LINEAR PROBLEMS

Elastic Layered Systems

Analytical Solutions—The geometrical domain of a layered system, i.e., the semi-infinite domain, and the regularity or "at-rest" conditions that exist at the boundaries make the problem particularly suited to analytical treatment. To obtain a solution to a layered system problem requires that the boundary condition be satisfied as well as the continuity conditions between the various layers. In principle, once the two-layer problem is solved, the methodology for solving the general multiple-layer system problem is established. It should be recognized that each layer is considered homogeneous and isotropic.

Since the original work by Burmister, the layered system problem has been analyzed extensively with a view toward obtaining numerical results and recasting the solution into a more general form. Satisfying the continuity conditions at the interface requires the solution of a number of algebraic equations, which include evaluation of infinite integrals. For most problems, involving more than two layers, the only practical method for obtaining a solution is to use a digital computer for the solution of the algebraic equations and the evaluation of the integrals. Practicing engineers should note that computer programs are now available for solving layered system problems formulated in accordance with the methodology first outlined by Burmister. The number of layers these programs can handle covers the range of all practical problems. Because these programs have been "debugged" and are operational, their use in routine design is a practical proposition. These programs provide information on stresses, strains, and displacements throughout the pavement system.

Numerical Solutions—There are fundamentally two numerical techniques currently in use for the solution of boundary value problems that are representative of the pavement system: the finite difference technique and the finite element technique. Of these two techniques, the latter appears to have the greatest potential. Details of the method have been discussed in the literature (see Appendix). The development of the finite element technique is directed toward a computational method, and the objective, from a practical standpoint, is to obtain a computer program that can efficiently solve pertinent boundary value problems. At the present time there are available, for general use, operational computer programs that can be used to solve anisotropic and nonhomogeneous linear elastostatic problems under axisymmetric conditions. It should be noted that, in comparison with the layered system formulation, the finite element technique is not subject to the restriction that each layer be homogeneous and isotropic. This property is of significance when it is necessary to include the effect of stress level on material properties. The cost of doing such analyses is minimal in terms of computer time, and the output provides information on stresses, strains, and displacements throughout the pavement section. Because of the rapidity with which these problems can be solved, it is relatively simple for the designer to study different designs.

In the analysis of thermal problems, it is assumed that the distribution of temperature can be obtained by solving the diffusion equation independently. This implies a decoupling of the temperature and stress effects. Finite element and finite difference programs are available for solving the diffusion equation to obtain the temperature distribution. Once the temperature distribution is obtained, its influence both in the form of a change in material properties or in introducing thermal stresses can be readily accounted for. Available finite element computer programs for the analysis of axisymmetric solids have the capability to analyze these temperature effects.

If inertial effects are neglected, the solutions to the moving load problem can be obtained by superposition of static solutions. Although it has been shown that inertial effects can be neglected at conventional highway traffic speed, it should be recognized that the finite element technique provides a means for analyzing dynamic problems and that operational computer programs are available.

Although computer programs for the axisymmetric case are generally available and are operational, it should be recognized that the actual problem belongs to the general three-dimensional class. Programs for solving such problems have been developed; however, they are not available for general use at the present time. The cost for conducting a three-dimensional analysis is far greater than that for an axisymmetric analysis. A possible technique for overcoming this disadvantage is to treat the problem in a general three-dimensional sense in the area in the close vicinity of the load and in an axisymmetric manner at a sufficient distance away from the load. Another possible alternative is to invoke St. Venant's principle and, instead of at-rest boundaries, to use the results from closed-form solutions of simple loading conditions at boundaries that can be located at much smaller distances from the loaded area than the at-rest boundaries. This would decrease the number of elements required to model the problem, reducing the computer time.

Thin Elastic Plate Formulation

As pointed out, the thin plate formulation is primarily used to consider the effect of edge loading conditions. The formulation has been used almost exclusively for the design of concrete pavements. It is a two-layer system, i.e., a plate and a supporting medium.

Analytical Solutions—The fundamental problem of concentrated load acting on a thin elastic plate resting on a foundation was solved by using the integral transform approach by Holl. Since that time numerous solutions have been developed for plates on various types of supporting media. The use of the thin plate theory to the design of pavements was first proposed by Westergaard; modifications based on theoretical and experimental considerations have been proposed by numerous investigators. Available analytical solutions for use in practice,do not consider nonhomogeneous and anisotropic material properties and do not adequately account for joints, cracks, and possible loss of support.

Numerical Solutions—The finite difference technique has been used fairly extensively in the analysis of plate problems. However, because of the difficulties in handling corners and because of the physically motivated formulation of the finite element method, most of the new developments in the analysis of plate problems are likely to be with the use of finite element techniques. Currently available operational computer programs can analyze plates resting on an elastic foundation under fairly general conditions. These include orthotropy nonhomogeniety in the soil and plate and localized loss of support.

Viscoelastic Layered Systems

As in elasticity, the boundary value problem consists of solving the governing differential equations subject to the appropriate boundary and initial conditions that represent the physical problem to be solved. The boundary conditions may be in the form of prescribed boundary stresses and displacements with regard to time and position, and the initial conditions indicate the conditions at time $t = 0$.

Because time occurs in both governing equations and boundary conditions, it is possible that the boundaries of prescribed stress and displacement may vary during the loading history. Such situations arise in the Hertz problem where with time the sphere indents the material, resulting in a change of the area of contact, thus varying the areas of prescribed displacement and stress. The total volume or surface area of the material may also change with time. Such situations occur in the case of crack propagation and in the case of ablating rocket propellents.

For pavement problems it is assumed that variations of surface and volume do not occur and that areas of prescribed deflection and stresses do not vary during the loading process. The formulation of the boundary value problems in viscoelastic systems is identical to that in elastic systems with the exception that stress-strain laws are time-dependent.

Analytical Solutions—Except for the simplest problems, solutions to boundary value problems in viscoelasticity rely on the extensive use of digital computers. The early emphasis in analyzing problems used the Laplace (or Fourier) transform technique, recognizing that a viscoelastic problem could be compared to some equivalent elastic problem in the transformed domain. The elastic solution has then to be inverted to obtain the required time-dependent solutions. This inversion can be extremely difficult and is only practical for simple representations of viscoelastic response in the form of differential operators. For realistic representations of viscoelastic response, it is necessary to use integral representations of the stress-strain-time relation of the materials based on experimentally measured creep or relaxation functions. In this case the Laplace transform technique leads to considerable difficulties, and it is more convenient to proceed directly with a spatial transform. The satisfaction of the continuity conditions at the interface leads to a set of simultaneous integral equations. Numerical solution of these equations and numerical evaluation of the spatial inversion result in the required solution. By using these techniques, we can apply experimentally obtained curves directly in the calculation. Solutions to the moving load problem, neglecting inertial effects, and a load applied periodically have been obtained by superposition of the static load solutions. The influence of the velocity and the period of application are of significance because of the time-dependent character of viscoelastic materials. Available solutions permit the computation of cumulative deformations as a function of load application and time. All currently available solutions are for the axisymmetric case.

For the practicing engineer it is necessary that operational computer programs be available for use in design. A number of computer programs using this approach have been developed. However, the dissemination of the information has been slow. It is necessary to present this information in a manner that will encourage its use by practicing engineers. This has not been done.

Numerical Solutions—The finite element technique has been applied to the solution of linear viscoelastic problems. Operational computer programs are now available, though their use has not yet become prevalent. It should be recognized that the computer time involved is far greater for solving viscoelastic problems than for solving elastic problems.

SOLUTIONS AND SOLUTION TECHNIQUES FOR NONLINEAR PROBLEMS

Although the importance of nonlinear analysis in the context of the total structural design of a pavement system has not been established, nonlinear problems must be solved if more realistic constitutive equations are to be used. It should be recognized that nonlinearity can also be introduced by large displacements, i.e., kinematic nonlinearity. Obtaining solutions to nonlinear problems depends almost exclusively on numerical techniques.

A number of ad hoc modifications of linear elastic theory have been used to solve "more realistic" boundary value problems. These modifications include a dependence of the modulus in a linear elastic analysis on stress. The problem is solved by iteration until a solution within some specified degree of convergence is obtained. Computer programs using the finite element technique to solve these ad hoc nonlinear problems are available, and their use does not present any more difficulty than does the use of linear programs.

It would appear at the present time that the finite element technique has the greatest potential for solving nonlinear problems. For illustrative purposes, consider the case of a nonlinear elastic constitutive law under the assumption of small strains.

Two methods of analysis commonly used for nonlinear problems are the incremental method and the Newton method with constant or variable slope.

The incremental load method breaks up the applied loads into n increments; a linear elastic solution is then sought for each increment, and the final solution is the sum of the increments. This procedure can use an incremental representation of the nonlinear elastic law along with a standard finite element computer algorithm for the solution of axisymmetric linear elastic boundary value problems (or any other method of solving linear boundary value problems). As an example, consider a layered system subjected to a uniformly loaded circular area. The pressure is divided into n increments and applied incrementally. The response of the system to the first increment can be obtained from the usual linear elastic theory. This will make possible the determination of principal strains throughout the layered systems.

From these known principal strain states the incremental moduli S_{ij} can be obtained for the next linear problem arising from the application of the second increment of applied pressure. The same procedure is repeated until the total load is applied. In terms of finite element nomenclature, the procedure requires that a new stiffness matrix be calculated for each load increment. By examining the range of principal strain states (both in magnitude and in ratios of components), we will be able to select the types and numbers of experiments required to characterize the nonlinear material behavior. This will in general vary with the particular problem (loading, layer thickness, or layer mechanical properties) and will require considerable cooperation between analysis and experimentation, perhaps ultimately linking the two into an integrated device for performing characterization and analysis.

The incremental approach can be used for conducting an elastoplastic analysis. It will be necessary to include a yield criteria to determine whether the material is in the plastic range. In addition, it will be mandatory to keep track of the stress path to determine whether loading or unloading is occurring. The same principles of incremental characterization and solution can be used for nonlinear viscoelastic solids. However, both the characterization of materials and the stress analysis algorithm are considerably more time-consuming.

The constant slope method places the nonlinear portion of the stiffness on the right side of the governing equation as a forcing function. The stiffness (slope) is the same for all iterations. The solution is then obtained by iteration. The variable slope method differs only in that the slope is updated after each iteration.

Finite element analyses based on the theorem of minimum potential energy are known to yield very accurate solutions for the displacements but frequently yield poor results for stresses. The few nonlinear solutions that have been attempted by using this method have not behaved well when the nonlinearity exceeds about 10 percent. However, the Hellinger-Reissner variational theorem provides a solution to these difficulties. Because both the displacements and stresses are included as primary variables, the resulting stresses are of a much improved accuracy and do not possess the spatial oscillations often found in displacement methods. The governing equations become the stress equations of equilibrium and the stress-strain law; whereas in the displacement formulation, the governing equations are the displacement equations of equilibrium. This is significant because the stress equations of equilibrium often are independent of the material properties, even for physically nonlinear materials. Thus, it is possible to obtain accurate solutions by using Newton's constant slope method with the above-mentioned method of analysis in a very few iterations. Displacement methods generally require many iterations that use more sophisticated models and that still fail at moderately high nonlinearities. Research done in this area indicates that the constant slope method has been found to be superior to the incremental load method when the mixed model is used and sufficiently convergent to give accurate results in a few iterations.

Another solution technique that may be of considerable significance in developing solutions for nonlinear boundary value problems is the technique of quasi-linearization developed by Bellman and his colleagues. The technique has been applied primarily to the solution of ordinary differential equations. In this context it is of considerable significance to problems in system identification, which are an important aspect of material characterization. The application of quasi-linearization to nonlinear partial differential equations, though limited, does indicate that it might be a powerful tool for the solution of such problems. It is claimed that the method has very good convergence and stability characteristics.

It is extremely important to recognize the problems associated with convergence when solution techniques are developed and used for nonlinear problems. It is not possible to check all facets of a computer program by comparison with known analytical results. Therefore, the use of solution techniques for nonlinear problems requires experience and judgment. It is important to realize the interdependence of the material characterization and solution techniques. If progress is to be made in the area of nonlinear analysis, research in these two areas will have to be closely coordinated. A great deal of work is needed in developing solutions for nonlinear problems before they can be used in practical designs.

SOLUTION TECHNIQUES—THE GENERAL PROBLEM

If, as is currently being advocated, a systems approach to the design of pavements is used, then it will be necessary to consider solution techniques for aspects of the problem other than the structural aspect. In this brief paper one can only call attention to certain tools that should be considered in an analysis of the pavement problem.

Dynamic programming in its application to optimization problems will be of considerable significance if optimum designs are to be developed. This will include the application of quasi-linearization techniques. One approach to the optimization of a design of the pavement system is to treat the system as an adaptive control process. In such processes we consider the problem of optimizing a process where our knowledge increases during the process. Dynamic programming is particularly suited to handling such problems. It should be recognized that dynamic programming is readily applicable to stochastic problems.

A great deal of work needs to be done in the application of these techniques to the pavement design problem.

FINAL REMARKS

In assessing where we are and where we are going with regard to solutions for boundary value problems, we should consider practical application and research separately. For routine structural design, operational computer programs for analyzing the axisymmetric linear elastostatic problem under very general conditions are available. The stresses, strains, and deflections in pavement sections can be determined and, in conjunction with various theories on modes of failure, e.g., fatigue, can be used in practical design. Because of the speed with which calculations can be performed, the designer can examine many combinations of materials and thicknesses. It can be concluded that the researcher has finished his task in the area of axisymmetric linear elastostatic problems and that the burden is on the designer to use this information.

In the area of linear viscoelasticity, sufficient theoretical work for axisymmetric linear analysis appears to have been completed for use in design. It remains now to disseminate the information to the practicing engineer and to provide guidance in its use.

From the research standpoint, there are a number of areas where further work appears necessary.

1. Conduct general three-dimensional linear elastic analysis. At the present time finite element computer programs are available to conduct such analyses. However, these programs cannot be considered operational in the sense that they can be used in routine design. Certain modifications will be necessary to tailor the programs for the efficient analysis of pavement systems.
2. Develop solution techniques and solutions for nonlinear problems. Considerable progress in this area has been accomplished; however, there has been little application to the pavement problem. The work to be done in the area cannot be done independently of nonlinear material characterization techniques.
3. Include stochastic and probabalistic concepts in analysis. The variability of materials and the nature of loads lead to the conclusion that complete analyses will require the inclusion of stochastic and probabalistic considerations in the solution of boundary value problems.
4. Develop solution techniques for the total pavement system problem. In considering the total pavement problem, researchers have spent a great deal of effort in developing various diagrammatic models of the pavement system. However, very little quantitative work has been done. It is now time to devote some effort in this direction. A possible avenue of future research is in the application of dynamic programming to the optimization of a pavement design system.

Before proceeding with research in various specific areas, we should direct the first effort toward integrating available solutions for boundary value problems with the other available information necessary for the development of a working structural design system and toward making it available for general use. There has been more progress in research than is generally realized. Once such a working design system is established, research needs for structural design can be better defined by sensitivity studies.

A major objective of engineering research is practical application. Therefore, it is imperative that the profession examine its needs for research in terms of significance in the context of the total system. For example, does it matter if materials are characterized as linear or nonlinear in the context of the variability in materials due to construction techniques? Has the structural design reached a degree of refinement that is sufficient in terms of the design of the total system? Perhaps this is a time for consolidation and integration of past scattered efforts and for evaluation of what is needed. However, we should not fall into the trap of trying to fit the needs to our capabilities. Rather we should expand our capabilities to fit the needs.

APPENDIX

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Editor's Note: The original paper included an extensive annotated bibliography, which has not been printed here. This bibliography is available in Xerox form from the Highway Research Board at cost of reproduction and handling. When ordering, refer to XS-38, HRB Special Report 126.