BASIC DATA PERTAINING TO FROST ACTION

HEAT TRANSFER AND TEMPERATURE DISTRIBUTION IN SOILS FOR TRANSIENT HEAT FLOW DUE TO CYLINDRICAL SOURCES AND SINKS

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Synopsis

Several methods of solving problems of transient temperature distribution and heat flow in the earth surrounding embedded heat sources and sinks in which the temperature is suddenly changed from that of the surrounding medium to a new value and maintained at this new level are presented.

Solutions of the differential equations for the temperature distribution and heat flow for the idealized case are evaluated in terms of dimensionless parameters. A numerical method is used in the study of the problem with real boundary conditions obtained from experimental observations. The method of electrical analogy is also presented as a rapid and accurate means of solving this problem. The thermal recovery (recuperation time) of the thermally disturbed soil is also studied and results shown.

The freezing or thawing rates of soils are a problem which can be studied by some of the methods used in the study of complex heat-transfer problems. The sudden change of thermophysical properties and the latent heat of transformation which results from the thawing and freezing of soils are not encountered, however, in the majority of heat-transfer problems, and therefore, appropriate modifications must be made in existing solutions to account for such phenomena. The electrical analogy method and the numerical methods are more readily adapted to include such phenomena than any other methods presently known.

It is not the purpose of this paper to solve any specific problem involving the freezing of soil but rather to discuss three basic methods of calculation which have been used in the study of transient temperature distribution in soils. In the examples used in this discussion the change of thermophysical properties of the soil which occur at the freezing temperature is not considered, since the work upon which this paper is based was of a preliminary nature and was concerned only with methods which would yield general solutions regarding temperature distribution in soils.

The exchange of heat between cylindrical heat sources or sinks and soil has attracted increasing interest lately by the recent attempts at a more rational solution of the problems involving the cooling of underground pipe lines and electrical cables and the use of the reverse-cycle refrigeration system for residence heating. Kafadar et. al. (1) presented a method for investigating the effects of freezing upon the temperature distribution in the soil around a cylindrical heat sink withdrawing heat from the soil at a constant rate. From the mathematical solution of the differential equation for this case they found a temperature gradient due to sensible heat withdrawal alone. Successive corrections of this gradient, which account for the latent heat withdrawal from the freezing zone and a final adjustment for the thermal conductivity of the frozen soil, lead to a temperature gradient compatible with the physical phenomenon.

Although the temperature gradients and the heat flow at any one point in the soil surrounding constant temperature heat sources or sinks are never invarient with respect to time; they will approach in time a near steady-state condition where their rates of change are very small. The time rates of change of the temperature distribution and heat flow, both of which may be extremely large during the initial part of the transient flow period, are of the utmost importance in many fields of engineering applications.

NOMENCLATURE

Symbol	Quantity	Units
A	Area	Ft^2
a	Tube radius	Ft
C	Heat capacity	$B\ F^{-1}$
C_E	Electrical capacitance	Farads
C_e	Electrical capacitivity	$Sec\ Ohm^{-1}\ Ft^{-3}$
C_T	Heat capacity	$B\ F^{-1}$
C_t	Volumetric heat capacity	$B\ Ft^{-3}\ F^{-1}$
c_p	Specific heat	$B\ lb^{-1}\ F^{-1}$
e	Napierian Logarithm base	Dimensionless
J_o	Bessel Function of the first kind and the zero order	Dimensionless
K	Thermal conductance	$B\ Hr^{-1}\ F^{-1}$
k	Thermal conductivity	$B\ Hr^{-1} Ft^{-1} F^{-1}$
m	Scale factor (in analogy)	$Ohm\ B\ Sec^{-1}\ F^{-1}$
N_o	Bessel Function of the second kind and the zero order	Dimensionless
n	Time factor (in analogy)	$Hr\ Sec^{-1}$
q	Rate of heat flow	$B\ Hr^{-1}$
R_E	Electrical resistance	Ohms
R_e	Electrical resistivity	Ohms Ft
R_T	Thermal resistance	$Hr\ F\ B^{-1}$
R_t	Thermal resistivity	$Hr\ Ft\ F\ B^{-1}$
r	Radial distance	Ft
t	Temperature	F
u	Variable of integration	Ft^{-1}
V	Electrical potential	Volts
v	Variable of integration (au)	Dimensionless
x	Distance	Ft
X	Radius parameter, $(\frac{r}{a})$	Dimensionless

Greek

α	Thermal diffusivity, $\frac{k}{\rho c_p}$	$Ft^2\ Hr^{-1}$
Φ	Time parameter, $\frac{\alpha\tau}{a^2}$	Dimensionless
Θ	Temperature difference parameter, $\frac{\theta}{\theta_o}$	Dimensionless
θ_o	Difference between initial uniform temperature and tube temperature	F
θ	Difference between initial uniform temperature and temperature at a given time and position	F
τ	Time	Hours
ρ	Density	$lb\ Ft^{-3}$

subscripts

1,2, -n refer to a particular region
21, etc. effect of region 2 on region 1, etc.

This paper deals with the transient temperature gradients in soil surrounding long, cylindrical heat sources or sinks, the temperature of which is suddenly changed from that of the soil and maintained constant at the new value. The paper also considers the changes in temperature gradients in the soil after the removal of the heat source or sink, that is, when the source or sink is no longer maintained at the constant temperature but is allowed to follow the soil temperature as the soil recovers towards its undisturbed thermal state.

Three different methods of approach to the problem are used. First, the exact mathematical solution of the differential equation of heat flow is evaluated. Unfortunately, this differential equation has been solved for only an idealized set of boundary conditions, but the complexity of this solution and its evaluation indicate that the solution for more practical boundary conditions would be too tedious to be practicable. Secondly, a numerical method of solution is used in which small finite time and space increments replace the corresponding quantities of differential magnitude. By means of this method practical boundary conditions can be embodied in the solution and the thermal recovery time of the soil can be investigated. The numerical method should be the most easily adapted to the study of frost penetration and thawing in soils. Thirdly, the method of the electrical analogy to the flow of heat is applied to the study of the problem. Again by this method real boundary conditions can be treated and the time for thermal recovery of the soil can be investigated.

Methods of Solution

In the application of each of the methods used in this paper certain general simplifying assumptions have been made. The soil surrounding the heat source or sink has been assumed to be homogeneous and isotropic. Although in only very few cases are soils reasonably homogeneous, the nature of the unhomogeneity is so unpredictible that such an assumption is advisable to enable the formulation of a manageable solution. The assumption of isotropicity seems to be generally sound. Heat flow in the soil is assumed to take place by conduction only, since in most dense, finely grained soils the effects of convection and radiation are negligible. Further, the effects of the change of thermophysical properties of the soil due to freezing or thawing and of migration of soil moisture due to the thermal gradients have been ignored. Although the change of the thermophysical properties would be exceedingly difficult to incorporate into the mathematical and electrical analogue methods, it could be done quite easily in the numerical method. The available data concerning moisture migration are inadequate, at present, to support an accounting for this effect; however, when such data become more plentiful the influence of moisture migration can be readily incorporated into the numerical method.

Mathematical Solution of Differential Equation

The differential equation describing the radial temperature history of the region surrounding a long cylindrical heat source is given by:

$$\frac{\partial \theta}{\partial \tau} = \alpha \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) \tag{1}$$

Carslaw and Jaeger (2) have solved this equation for the following boundary conditions: (1) The heat source consists of an infinitely long cylinder of radius a; (2) the surrounding medium is homogenous, of infinite extent in all directions, and at a uniform initial temperature of zero; and (3) at time τ equals zero the cylinder is suddenly raised to the temperature θ_0, after which it is maintained at this temperature. For these boundary conditions the solution of the differential equation was found to be

$$\theta = \theta_0 + \frac{2\theta_0}{\pi} \int_0^\infty e^{-\alpha u^2 \tau} \frac{J_0(ur)N_0(ua) - N_0(ur)J_0(ua)}{J_0^2(ua) + N_0^2(ua)} \frac{du}{u} \tag{2}$$

Using this equation for the temperature distribution in the general equation for radial heat flow,

$$q = -k2\pi r\left(\frac{\partial\theta}{\partial r}\right) \tag{3}$$

Carslaw and Jaeger found the heat flow across the surface of the cylinder to be

$$q = \frac{4\theta_0 k}{a\pi^2}\int_0^\infty e^{-au^2\tau}\frac{du}{u[J_0^2(ua) + N_0^2(ua)]} \tag{4}$$

In rearranging Equations 2 and 4 for evaluation Gemant (3) introduced the following dimensionless parameters:

$$\Theta = \frac{\theta}{\theta_0}, \qquad X = \frac{r}{a}, \qquad \Phi = \frac{\alpha\tau}{a^2}, \qquad v = au,$$

Substituting these parameters into Equations 2 and 4 the equation for the temperature distribution becomes

$$\Theta = 1 + \frac{2}{\pi}\int_0^\infty e^{-\Phi v^2}\frac{J_0(xv)N_0(v) - J_0(v)N_0(xv)}{J_0^2(v) + N_0^2(v)}\frac{dv}{v} \tag{5}$$

and the equation for heat flow becomes

$$\frac{q}{\Theta_0 k} = \frac{8}{\pi}\int_0^\infty e^{-\Phi v^2}\frac{dv}{v[J_0^2(v) + N_0^2(v)]} \tag{6}$$

For numerical evaluation the integrals of Equations 5 and 6 are broken down into three parts:

$$\int_0^\infty = \int_0^{v_1} + \int_{v_1}^{v_2} + \int_{v_2}^\infty \tag{7}$$

It has been shown by Gemant that if a value of v_1 be chosen such that $\Phi v_1^2 \ll 1$ and $xv_1 \ll 1$ then the first integral of Quation 5, when reduced to the form of Equation 7, becomes

$$\int_0^{v_1} e^{-\Phi v_1^2}\frac{J_0(xv)N_0(v) - J_0(v)N_0(xv)}{J_0^2(v) + N_0^2(v)}\frac{dv}{v} = \frac{ln.\ x}{ln.\ v_1 - 0.116} \tag{8}$$

It was also shown that by choosing v_2 such that

$$v_2 \gtrsim \frac{3}{\sqrt{\Phi}}$$

the third integral of Equation 5, when reduced to the form of Equation 7, can be neglected. Equation 5 therefore has been reduced to

$$\Theta = 1 + \frac{lnx}{lnv_1 + 0.116} + \frac{2}{\pi}\int_{v_1}^{v_2} e^{-\Phi v^2}\frac{J_0(xv)N_0(v) - J_0(v)N_0(xv)}{J_0^2(v) + N_0^2(v)}\frac{dv}{v} \tag{9}$$

The integral of Equation 9 can be evaluated numerically between the finite limits v_1 and v_2 .

In a similar manner it has been shown that by choosing

$$\Phi v_1^2 \ll 1, \quad v_1 \ll 1, \text{ and } v_2 \lesseqgtr \frac{3}{\sqrt{\Phi}}$$

Equation 6 can be reduced to the form

$$\frac{q}{\Theta k} = \frac{-2\pi}{\ln v_1 - 0.116} + \frac{8}{\pi}\int_{v_1}^{v_2} e^{-\Phi v^2} \frac{dv}{v[J_0^2(v) + N_0^2(v)]} \tag{10}$$

which also can be evaluated numerically.

The transient temperature distribution in the medium surrounding a line source or sink can be determined from Equation 9 and the heat flow at the surface of the source or sink from Equation 10.

Numerical Method of Solution

Numerical methods have been used in the solution of engineering problems for many years. Much has been written on the subject of these methods applied to special fields of interest (4), (5). The method applied to heat conduction as presented by G. M. Dusinberre (6) consists of dividing the thermal system into a number of reference regions and establishing simultaneous and independent heat balances between each region and its adjoining regions. This application of the method is based upon the following three assumptions: (1) Negligible error is introduced by using the temperature change of a central point in any region in computing the change of heat stored in the region due to this temperature change, (2) a time interval can be chosen sufficiently small that there is negligible error in using the initial temperature gradient between central points of adjoining regions in computing the heat flow between these regions during this time interval, and (3) during this time interval any region is affected only by those regions adjoining it.

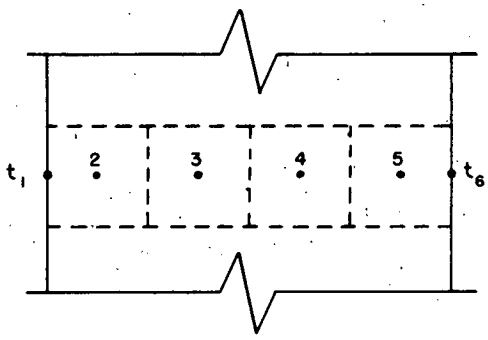

Figure 1A. Thermal System for One Dimensional Heat Flow Through a Slab.

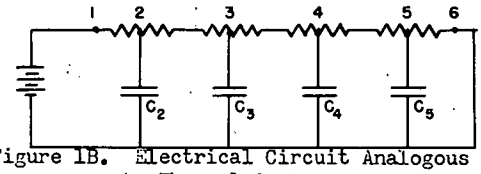

Figure 1B. Electrical Circuit Analogous to Thermal System.

The rate of heat flow between any two adjoining regions is dependent upon the overall transmittance, K, and the temperature difference between the two regions. The rate of heat flow from a region 2 into a region 1 may be expressed as

$$q_{21} = K_{21}(t_2 - t_1) \tag{11}$$

Similar equations may be written for the flow into region 1 from all other adjoining regions, so that the total rate of heat flow into region 1 becomes

$$q_1 = K_{21}t_2 + K_{31}t_3 + \cdots + K_{n1}t_n - \sum_{n=2}^{n} K_{n1}t_1 \tag{12}$$

According to assumption 2 above, Equation 12 gives the rate of heat flow into region 1 during the time interval $\Delta\tau$. During this time interval the temperature of the midpoint of region 1 changes from t_1 to t_1', hence, according to assumption 1, the rate of heat storage in region 1 during this time interval can be given as

$$q_1 = \frac{C_1(t_1' - t_1)}{\Delta\tau} \tag{13}$$

Since heat is stored in a region as a consequence of the net heat flow into the region, Equations 12 and 13 can be equated to give:

$$t_1' = \frac{K_{21}\Delta\tau t_2}{C_1} + \frac{K_{31}\Delta\tau t_3}{C_1} + \cdots + \frac{K_{n1}\Delta\tau t_n}{C_1} + \left[1 - \sum_{n=2}^{n}\frac{K_{n1}\Delta\tau}{C_1}\right]t_1 \tag{14}$$

as an expression for the temperature, t_1', of the midpoint of region 1 after the time interval $\Delta\tau$. In equation 14 it is apparant that the coefficients of the temperatures of various regions are constants depending upon the physical constants of the particular problem. These coefficients are known as weighting factors and Equation 14 can be rewritten as

$$t_1' = F_{21}t_2 + F_{31}t_3 + \cdots + F_{n1}t_n + F_{11}t_1 \tag{15}$$

If F_{11} were chosen to be negative, an erroneous oscillation or divergence in the calculated temperature would occur, since the new temperature of point 1 would depend upon its old temperature in a negative sense. The criterion for convergence must therefore be

$$F_{11} \geqq 0 \tag{16}$$

or

$$\left[1 - \sum_{n=2}^{n}\frac{K_{n1}\Delta\tau}{C_1}\right] \geqq 0 \tag{17}$$

The maximum value of $\Delta\tau$ permissible must therefore be

$$\Delta\tau_{max.} = \frac{C_1}{\sum_{n=2}^{n} K_{n1}} \tag{18}$$

If the temperature distribution in a medium in which heat conduction is taking place is known at any time, τ, the temperature distribution at a time $\tau + \Delta\tau$ can be found by subdividing the medium into appropriate regions and solving Equation 15 for each region. If any region undergoes a process involving latent heat it may be taken into account by adding a latent heat term, q_L, to Equation 12 which appears as the added term, $\frac{q_L\Delta\tau}{C_1}$, in Equations 14 and 15. If a region undergoes a change of thermophysical properties the weighting factors involving this region must of course be changed for subsequent steps of the calculation.

Method of Electrical Analogy

The mathematical laws expressing the conduction of heat in solids and the flow of current in certain noninductive circuits are identical, therefore, it is possible to construct an electrical circuit in which the flow of current is analogous to the flow of heat in a solid and the potential distribution is analogous to the temperature distribution in the solid. The time factors in such an analogous electrical circuit can be so adjusted at will that a thermal process can be reproduced electrically in much greater or less time than would be required for the actual thermal process to take place. For this reason the electrical-analogy method is to be preferred for the study of many heat-transfer problems which involve long time periods.

For a slab of infinite length such as is shown in Figure 1A, the temperature history can be described by the differential equation

$$\frac{\partial t}{\partial \tau_t} = \alpha\frac{\partial^2 t}{\partial x_t^2} \tag{19}$$

In Equation 19

$$\alpha = \frac{k}{\rho c_p} = \frac{1}{\frac{1}{k}\rho c_p}$$

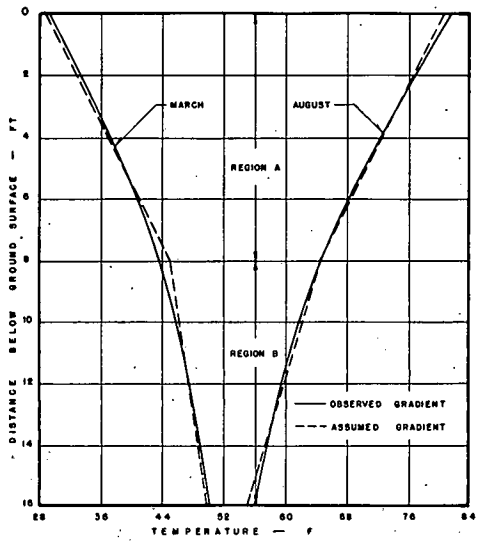

Figure 2. Temperature Gradients in Ground.

If we let $\frac{1}{k} = R_t$ (thermal resistivity) and $\rho c_p = C_t$ (volumetric heat capacity) then $\alpha = \frac{1}{R_t C_t}$ and Equation 19 becomes

$$\frac{\partial t}{\partial \tau_t} = \frac{1}{R_t C_t} \frac{\partial^2 t}{\partial x_t^2} \tag{20}$$

For an electrical circuit with uniformly distributed resistance and capacitance such as is shown in Figure 1B the voltage history can be described by the differential equation

$$\frac{\partial V}{\partial \tau_e} = \frac{1}{R_e C_e} \frac{\partial^2 V}{\partial x_e^2} \tag{21}$$

The similarity of the flow of electricity and heat can be seen by comparison of Equations 20 and 21. The heat capacities of the four elements 2, 3, 4, and 5 of Figure 1A are represented electrically by the condensers C_2, C_3, C_4, and C_5 in Figure 1B. Similarly, the thermal resistivity between any two points of Figure 1A is represented by the corresponding section in Figure 1B.

A circuit can be constructed in which the values of the resistance and capacitance are numerically equal to those of the corresponding thermal quantities. In such a circuit the transient voltage changes occur in the same time in seconds as the analogous temperature changes occur in hours. Paschkis (7), has suggested that the electrical analogy can be made more versatile in the following manner. If it is desired for the transient time factors in the electrical circuit to be different from those in the thermal circuit the τ_t appearing in Equation 20 may be reduced by a factor, n, such that $\tau_t = n\tau_e$. The denominator of the right side of Equation 20 must also be reduced by the same factor, n. Equation 20 therefore becomes:

$$\frac{\partial V}{\partial \tau_e} = \frac{1}{\frac{R_t}{n} C_t} \frac{\partial^2 V}{\partial x^2} \tag{22}$$

It is possible that the resistance and capacitance units which correspond in magnitude to the desired thermal properties may not be obtainable. The constant m may be introduced into the right side of Equation 22 in such a manner that the equation is not changed.

$$\frac{\partial V}{\partial \tau_e} = \frac{1}{\left(\frac{R_t m}{n}\right)\left(\frac{C_t}{m}\right)} \frac{\partial^2 V}{\partial x^2} \tag{23}$$

It is evident that Equations 23 and 21 are identical if the two conditions

$$R_e = \frac{R_t m}{n} \tag{24}$$

$$C_e = \frac{C_t}{m} \tag{25}$$

are fulfilled. By proper selection of the magnitude of n and m to satisfy the condition $\tau_t = n\tau_e$ as well as Equations 24 and 25 a convenient time increment and feasible sizes of resistors and condensers may be obtained. The voltage V can be any convenient value. It must only be remembered that the total applied voltage V represents the over-all temperature difference and that the voltage at any point in the circuit represents the temperature excess at the corresponding point in the thermal system.

Application of the Methods

The three methods of solving heat-transfer problems discussed in general terms above will now be applied to the study of a typical problem. It is desired to investigate the transient temperature distribution in the soil surrounding a single or a group of four horizontal tubes embedded 8 ft. below the ground surface. These tubes may well represent the ground coil of a reversed-cycle refrigeration system. Two conditions will be investigated: a winter condition in which the ground is initially at 45 F. and the coil is suddenly changed to 20 F., and a summer condition in which the ground is initially at 64.5 F. and the coil is suddenly changed to 110 F. The thermophysical properties of the soil are selected from data of Kersten (8). These data are apparently the best available at the present time.

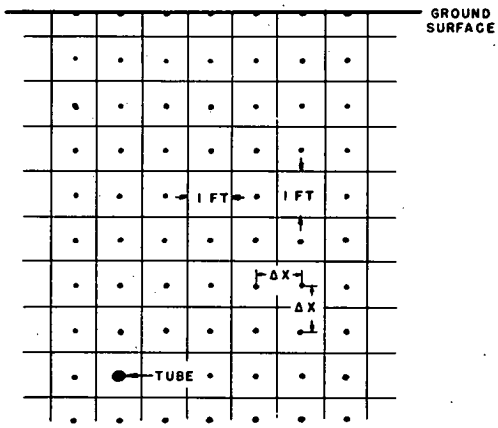

Figure 3. Reference Network Used With The Numerical Method.

The mathematical solution of the differential equation which incorporates idealized boundary conditions and not the actual boundary condition of the problem is evaluated to be used as a reference solution and as a check on the accuracy of the electrical analogue. Even though this solution does not represent the actual problem specified above, it could be applicable if the tubes were embedded 15 to 20 ft. deep where the ambient soil temperature is very nearly uniform and the influence of the ground surface boundary conditions are negligible. Equations 9 and 10 for the temperature distribution and heat flow respectively are evaluated by graphical integration.

The study of the problem incorporating the actual boundary conditions is made by means of the numerical method of solution. From experimental data on ground-temperature variation throughout the year (9), the maximum and the minimum ground temperatures were found to occur in August and March respectively, as shown in Figure 2. The ground temperature passes through an annual cycle between these two gradients, but the extremes have been used in order to arrive at a conservative solution of the problem. For the short period of time considered in this investigation (maximum 12 days) the change in ground temperature gradients is negligible; hence to simplify the calculations the gradients shown in Figure 2 are assumed to be steady state gradients.

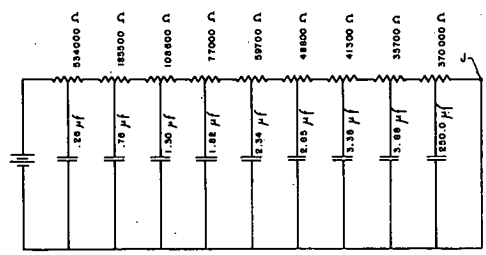

Figure 4. Electrical Analogy Circuit For 13/16-in. Diameter Tube.

From the equation for steady-state heat conduction through a slab

$$q = kA\frac{dt}{dx} \tag{26}$$

it can be seen that the temperature gradient $\frac{dt}{dx}$ must vary inversely as the conductivity k. Therefore if the gradients shown in Figure 2 are considered steady-state gradients, then the conductivity of the soil must increase with depth. In order to further simplify the calculations the actual gradients of Figure 2 have been approximated by the two

dashed straight-line segments shown, thus necessitating only two layers of different conductivity. The ratio of the approximating gradients in layers A and B are 1. 8 to 1 for the summer gradient and 4 to 1 for the winter gradient. Hence, the ratio of the conductivities of layers A and B respectively must be 1 to 1. 8 for the summer and 1 to 4 for the winter. From the data of thermal properties of soils (8) the following values were chosen arbitrarily for the purpose of this example:

	Winter Conditions	Summer Conditions
Region A	$k = 0.25$	$k = 0.28$
	$\rho = 83$	$\rho = 83$
	$c_p = 0.18$	$c_p = 0.18$
Region B	$k = 1.00$	$k = 0.50$
	$\rho = 135$	$\rho = 100$
	$c_p = 0.22$	$c_p = 0.20$

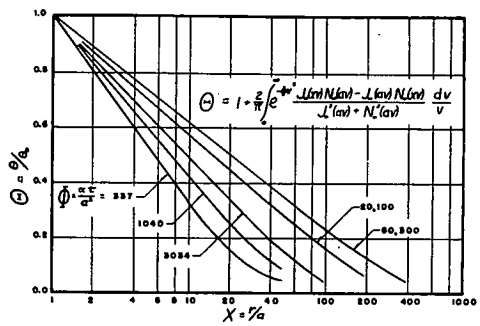

Figure 5. Temperature Distribution For Idealized Case.

The reference regions used in this numerical method have been formed by superimposing a square grid onto the crosssection of the soil perpendicular to the tube using the center point in each square as a reference point (Fig. 3). With 1-ft. -square grids the thermal conductance between the reference points in layer A for the winter condition is

$$K = \frac{kA}{\Delta x} = \frac{.25(1)}{1} = .25\ BHr^{-1}F^{-1}$$

The heat capacity of each section is

$$C = c_p \rho V = .18(83)(1) \approx 15\ BF^{-1}$$

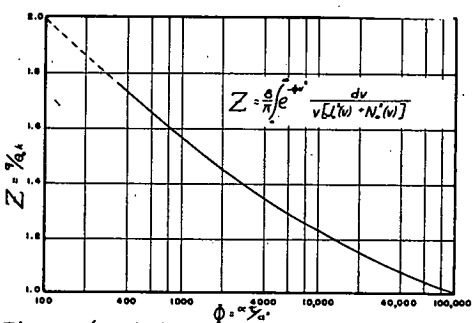

Figure 6. Rate of Heat Flow at Tube Surface.

From Equation 18 the limiting value of $\Delta\tau$ is 15/4(0. 25) = 15 hr. Similarly for layer B, K = 1, C = 30, and $\Delta\tau$ = 7. 5 hr. Since it is desirable to have the same $\Delta\tau$ in both layers and also for this value to be a multiple of 24 hr., a time interval of 6 hr. is chosen. Using $\Delta\tau$ = 6 hr. the weighting factors for the temperature of each point in layer A surrounding the point in question is

$$F_{n1} = \frac{K_{n1}\Delta\tau}{C_1} = \frac{.25(6)}{15} = .1$$

The weighting factor for the point itself is $F_{11} = 1 - \Sigma F_{n1} = 1 - 4(.1) = .6$. Similarly for layer B the weighting factors are found to be $F_{n1} = .2$ and $F_{11} = .2$. Weighting factors for the points lying on the plane of discontinuity between layers A and B, the midplane, require special consideration. The factor F_{11} weighting the influence of the temperature of one point on the midplane upon another point on the midplane is based on the arithmetic mean of the physical properties of the two layers. For two points on the midplane then

$$F_{n1} = \frac{[.25 + 1.00]6}{15 + 30} = .167$$

$$F_{11} = 1 - .10 - 2(.167) - .20 = .366$$

The initial temperature distribution in the soil surrounding the tube is known from Figure 2. For the first step in the calculations the grid point representing the tube is

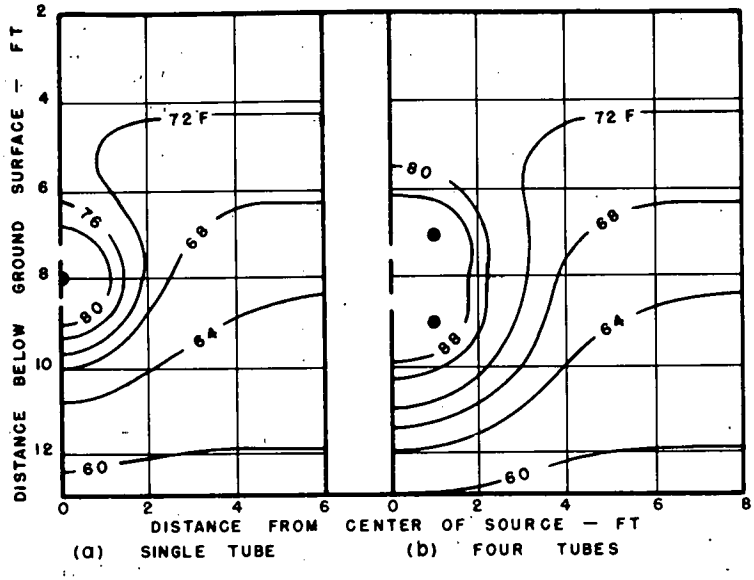

Figure 7. Temperature Distribution After Three Days Operation of 110 F Source.

assigned the temperature 20 F. which is held constant throughout the remaining steps. The ground-surface temperature is fixed at a constant value of 29 F. although it could be varied at will. Equation 15 is now calculated for each grid intersection to find the new temperature distribution after the time increment $\Delta\tau = 6$ hr. This process is continued for 12 steps to find the temperature distribution after three days. To study the thermal recovery of the ground after the heat sink is removed, calculations are continued as above using as a starting distribution the calculated distribution existing at the time the heat sink is removed. The grid intersection representing the tube is now no longer maintained at 20 F. but is allowed to change as any other point and its temperature calculated at each step.

The method of the electrical analogy as used here serves two purposes. The data obtained can be compared directly with the results of the mathematical solution as a check on the electrical analogue results. Also, the data can be interpreted in such a manner as to afford a check between it and the numerical method of solution.

The electrical analogue circuit itself is representative of the idealized boundary conditions which were assumed in the mathematical solution. The primary interest is the zone within 2 ft. of the heat source and from the results of the mathematical solution it is found that there is no disturbance of the temperature beyond 8 ft. from the source in the 12 days considered here. Consequently, the zone from the source to a radius of 2 ft. from the source is divided into eight concentric sections each 0.25 ft. wide. The zone from 2 ft. to 8 ft. radius from the source is lumped into one section. Normally one second in the electrical analogue represents one hr. in the thermal system, however, it is desirable to let one second in the analogue represent 24 hr. in the thermal system, hence,

$$\tau_t = n\tau_e$$

$$24 \text{ Hrs.} = n \times 1 \text{ Sec.}$$

$$n = 24$$

It is known that the amount of heat conduction through a hollow cylinder is given by

$$q = -\frac{2\pi kL}{ln\,\frac{r_2}{r_1}}(t_2 - t_1) \tag{27}$$

This equation may be written as

$$q = -\frac{1}{R_T}(t_2 - t_1) \tag{28}$$

Comparing Equations 27 and 28 it is seen that the thermal resistance, R_T, is given by

$$R_T = \frac{\ln \frac{r_2}{r_1}}{2\pi k L}$$

The thermal resistance for the first section in the A region for the winter condition is found to be

$$R_T = \frac{\ln \frac{.25}{.03368}}{2\pi(.28)(1)} = 1.14\ HrFB^{-1}$$

Based on Equation 24,

$$R_T \frac{m}{n} = R_E \quad \text{or} \quad \frac{1.14m}{24} = R_E$$

Choosing $m = 11.3 \times 10^6$ gives $R_E = 534000$ which is a reasonably sized maximum resistance unit. Checking this value of m to find what maximum size condenser is required we find that

$$C_T = c_p \rho V = c_p \rho \pi (r_2^2 - r_1^2)\, L = .18(83)\, \pi(.25^2 - .03368^2) = 2.88\ BF^{-1}$$

and on the basis of Equation 25

$$C_E = \frac{C_T}{m} = \frac{2.88}{11.3 \times 10^6} = .26 \times 10^{-6} \text{ farads} = .26\ \mu f$$

This results in condensers of reasonable size. Continuing in this manner using m = 11.3 x 10^6 and n = 24, the sizes required for the remaining resistors and condensers

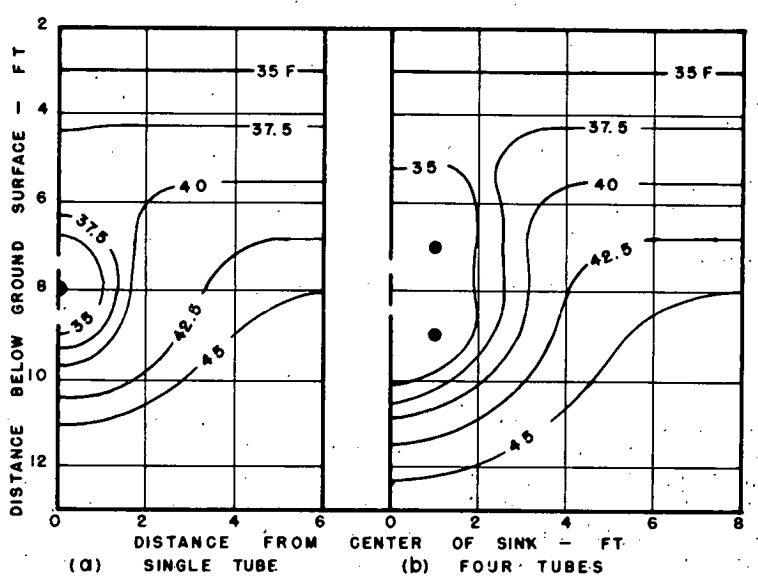

Figure 8. Temperature Distribution After Three Days Operation of 20 F Sink.

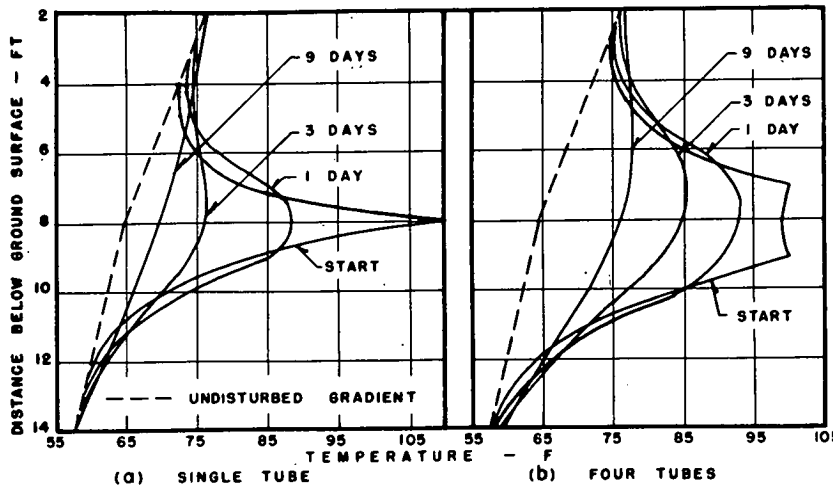

Figure 9. Thermal Recovery of Soil After Three Days Operation of 110 F Source.

are determined. The electrical analogue circuit is shown in Figure 4.

In order to use the same electrical circuit for the layers A and B, which have different thermophysical properties, the time factor must be different for the two regions. Since the electrical units are the same, the time constant must be changed by the same ratio as the thermal units.

$$\frac{R_{eB}\, C_{eB}\, n_B}{R_{eA}\, C_{eA}\, n_A} = \frac{R_{tB}\, C_{tB}}{R_{tA}\, C_{tA}}$$

or

$$n_B = n_A \left[\frac{R_{tB}\, C_{tB}}{R_{tA}\, C_{tA}}\right] = n_A \left[\frac{\frac{1}{k_B}(\rho_B\, c_{pB})}{\frac{1}{k_A}(\rho_B\, c_{pA})}\right] = 24 \left[\frac{\frac{1}{.50}(100 \times .20)}{\frac{1}{.28}(83 \times .18)}\right] = 18$$

Hence, from Equation 24

$$\tau_{eB} = \frac{\tau_t}{n_B} = \frac{24 \text{ Hr}}{18 \text{ HrSec}^{-1}} = 1\tfrac{1}{3} \text{ Sec.}$$

The same circuit thus represents the layer B if the data are interpreted such that 1-1/3 seconds in the electrical circuit are equivalent to 24 hr. in the thermal system.

The thermal recovery of the soil after removal of the heat source is investigated by removing the applied voltage and allowing the condensers to discharge to ground potential at point J in Figure 4.

To compare the results of the electrical analogue with those of the numerical method, the data obtained for layers A and B must be combined graphically. The data corresponding to region B is combined by smooth curves with the data of region A and the resulting distribution is then superimposed graphically upon the assumed steady-state gradient which existed in the undisturbed soil.

Discussion

The results obtained by using the methods previously outlined are presented in Figures 5 to 11. Figure 5 shows the evaluation of Equation 9 for the temperature distribution around a cylindrical heat source embedded in a homogeneous medium initially at uniform temperature. These results are presented in terms of the dimensionless parameters and are applicable over a wide range of the variables involved. Similarly Figure 6 presents the data obtained by the evaluation of Equation 10. This curve represents the heat-transfer rate across the surface of a single tube. The rate of heat flow is theoretically infinite at the initial moment, but as seen, it decreases rapidly to

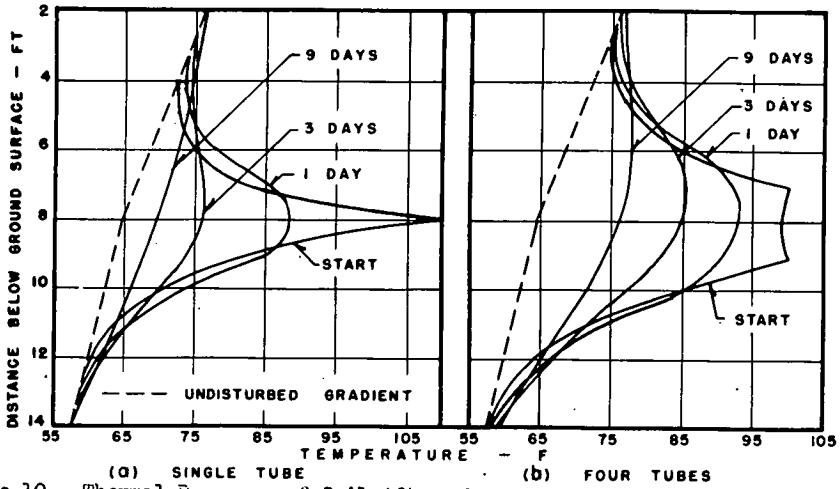

Figure 10. Thermal Recovery of Soil After Three Days Operation of 20 F Sink.

a finite value. The results obtained by using the numerical method for finding the temperature distribution around a single tube placed 8 ft. under the ground surface and operated as a heat source at 110 F. for three days are presented in Figure 7. The figure also shows the temperature distribution around a group of four tubes when operated in the same manner. The extreme right portion of each part of the figure shows the horizontal extent of the distance of influence, and the temperature distribution here is that of the original undisturbed soil. Similarly Figure 8 shows the results corresponding to conditions of Figure 7 for the case of a heat sink operated at 20 F. for three days. Figures 9 and 10 present the results of the numerical calculations of the thermal recovery of soil after removing the sources and sinks respectively. Part (A) of each figure shows the temperature distribution along the vertical center line of the single tube during thermal recovery and part (B) the corresponding distribution for the group of four tubes. The initial gradient in each case is that which was found to exist after operating the sources and sinks for three days. It is seen that after operating the sources and sinks for three days, nine days are required for the soil at the depth of the tubes to return to within about 3 F. of the original undisturbed temperature. It was also found that after one and two days of operation of the heat sources and sinks a recovery time three times as long as the operation time was required.

Figure 11 shows a comparison between the results obtained from the electrical-analogy circuit and the mathematical solution for a 13/16-in. diameter tube. As can be seen from the figure the results of the two methods are in good agreement.

A comparison of the results obtained from the numerical method and the electrical-analogy method for the region close to the tube showed that for corre-

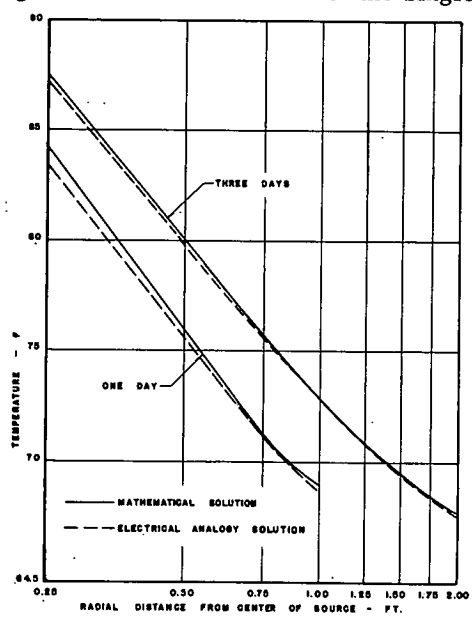

Figure 11. Temperature Gradient in Soil Around 13/16-in. Source Maintained at 110 F. For One Day and For Three Days.

sponding points in the system the electrical method gave a slower response to a change of tube temperature than did the numerical method. The more rapid response observed in the results of the numerical method is actually an error due to a violation of the first assumption upon which the method is based. The first assumption as given previously implies that the temperature of the central point in any region is the average temperature of the entire region. In the region that includes the tube this is not true, especially during the time immediately after the tube temperature is changed if the volume of the region is large compared to that of the tube. Since this assumption implies that the tube completely fills its own region the accuracy may be increased by choosing space increments near the tube much smaller than the 1 ft. used in this example. A check made using space and time increments much smaller showed that the temperature at a point 1 ft. away from the source approached more closely the temperature obtained by the electrical analogue.

In this respect the advantages of the use of cylindrical coordinates for this particular problem are worth mention. By using cylindrical coordinates the space increments near the source or sink can be made small and increasingly larger further away from the source. This will improve the accuracy to a great extent but at the expense of more computational labor.

The methods discussed in this paper should prove to be valuable tools in the study of the rates of frost penetration and thawing in soils. The application of all of the methods is considerably simplified when large, plane heat sources and sinks are concerned. Mathematical solutions are advantageous in only those cases in which the boundary conditions are simply defined. The electrical-analogy method should prove very useful when one geometrical system is to be studied under several different sets of boundary conditions and thermophysical properties. If a single study incorporating complex boundary conditions is to be made, then in general, the numerical method should prove to be the most useful.

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