# Mathematics of Isothermal Water Conduction in Unsaturated Soil

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### Introductory Remarks by the Chairman

"I believe the day must come when the biologist will-without being a mathematician-not hesitate to use mathematical analysis when he requires it." This statement was made by Karl Pearson in the January 17, 1901, issue of "Nature." The day has come, indeed, not only for the biologist but also for his brother the soil scientist, as the author proves in his excellent contribution to this symposium. It may be appropriate to supplement Pearson's words by the following remarks which Dr. Johnson made in the fourteenth "Rambler," May 5, 1750: "The mathematicians are well acquainted with the difference between pure science, which has to do only with ideas, and the application of its laws to the use of life, in which they are constrained to submit to the imperfections of matter and the influence of accident."

• THE RATE of water movement in soil is of considerable importance in agriculture and is being actively investigated. The physical properties of soils influence water movement in two important ways. The sizes and structural arrangement of the soil particles determine the space configuration through which water moves, and the interaction between the soil and the soil water gives rise to water-moving forces. In soils in which the pores are completely filled with water, the fluid is a single phase. Forces originating from gravity, including the pressure gradient, determine the flow. The saturated flow process is described mathematically by Darcy's law and Laplace's equation.

It is more often the case in agricultural soils that the pores are not completely filled with water and the potential of the water depends not only upon its position in the gravitational field, but also upon adsorptive force fields associated with interfacial boundaries. The imbibition or retention of water by soil is due to a number of mechanisms which can be only briefly mentioned here. The total potential can be conveniently expressed as the sum of a gravitational potential, an osmotic potential, and a capillary potential. The osmotic term is usually omitted in considering water movement because the solutes generally move more readily than the water with respect to the soil. Of most concern is the capillary potential which here includes all the contributions from the various sorption mechanisms. These mechanisms are discussed in detail elsewhere and will not be elaborated upon. The capillary potential is often expressed in terms of the soil suction, or soil-moisture tension. The soil suction is defined as the pressure difference required across a permeable membrane to produce hydraulic equilibrium between soil water that is subject to surface-force action and free water that is not subject to surface-force action.

Permeable membranes are usually employed in the measurement of this suction. In the tensiometer, a manometer is connected to a porous cup placed in the soil. The water in the manometer is connected to the soil water through the porous cup. The soil suction is frequently expressed in terms of the height of a column of water which can be supported by the soil water and in equilibrium with it. This instrument cannot normally be used for suctions above about 800 cm of water, and for the higher suctions, Richards (19, 20) has developed the pressure membrane and pressure plate apparatus. In these apparatus, the soil is placed in a chamber on a porous membrane and air pressure applied to the chamber. Water flows between the soil and free water outside the chamber until equilibrium is attained. The pressure in the chamber is a measure of the suction and the water content can be obtained, thus giving a functional relationship between water content and potential. A knowledge of the suction as a function of water content is extremely useful in understanding the retention and transmission of water by any given soil. Figure 1 shows the water retention plotted as a function of suction for three representative soils, Chino clay, Indio loam, and Pachappa sandy loam.

The freezing point depression of the soil water has been used to measure the potential, and methods for measuring the vapor pressure of the soil water which are now being developed should prove extremely valuable in potential measurements. These measurements include the osmotic term.

### THE FLOW EQUATION

Water in the soil moves in response to potential differences and in the direction of decreasing potential. The basis of the theory of unsaturated flow is the assumption that the volume flux of water per unit area perpendicular to the direction of flow

is directly proportional to the potential gradient. This gives an expression analogous to Darcy's law for saturated flow

soils,

$$= \mathbf{k} \nabla \mathbf{\Phi}$$
 (1)

where v is the volume flux and  $\phi$  the potential. Mathematically, this expression differs from Darcy's law only in that the quantity k, the capillary conductivity or unsaturated permeability, is not constant but is a function of the soil-water content or the suction.

The movement of water must also obey the principle of conservation of matter, which is expressed in the equation of continuity

$$\frac{\partial \theta}{\partial t} = -\nabla \cdot v \tag{2}$$

where  $\theta$  is the water content on a volume basis and t is the time. Combining Eqs. 1 and 2 gives the flow equation

$$\frac{\partial \theta}{\partial t} = -\nabla \cdot k \nabla \phi \tag{3}$$

It is usually convenient to separate the total potential into the gravitational and capillary potentials as in Eq. 4

 $\frac{\partial \theta}{\partial t} = -\nabla \cdot k \nabla \psi + \frac{\partial k}{\partial z}$ (4)

where  $\psi$  is the suction.

As in the case of saturated flow, the description of the water movement in a soil system becomes a matter of establishing the boundaries of the system, specifying the conditions at these boundaries and at the initial time, and then solving the partial differential Eq. 4 subject to these boundary and initial conditions. To do this, it is necessary to know the functional relationships between the water content, the suction, and the capillary conductivity. This requires the use of single-valued functions, which can be obtained if the water content always decreases or increases, but which may have to be approximated otherwise because of hysteresis in the water content-suction curve.



water content for three representative

and Chino clay. Data is for the desorption

cycle.

Indio loam, Pachappa sandy loam,

The problem is further complicated if inhomogeneities or temperature gradients are present, since the suction may then also depend upon position and temperature. This discussion will be restricted to isothermal homogeneous systems. Then either  $\psi$  or  $\theta$  may be eliminated from Eq. 4 and the other used as the dependent variable, with x, y, z, and t as the independent variables. If the potential is eliminated from Eq. 4

$$\frac{\partial \theta}{\partial t} = \nabla \cdot D\nabla \theta + \frac{\partial k}{\partial z}$$
(5)

where

$$D(\theta) = -k \left(\frac{\partial \psi}{\partial \theta}\right) \tag{6}$$

For horizontal flow the last term on the right hand side of Eq. 5 is omitted and Eq. 5 is identical in form with the well-known diffusion equation, and  $D(\theta)$  is called the diffusivity. This does not, however, imply that the mechanism of water movement is diffusion in the same sense as diffusion in gases, liquids, or solids due to random molecular motion. Several mechanisms are probably involved, depending upon the water content and the nature of the soil particles and their arrangement. Vapor movement is included in the equation to the extent that the water vapor is in equilibrium with the liquid water in the surrounding pore. Attempts are being made to elucidate these mechanisms through studies of the dependence of k and D upon water content, temperature, and soil properties, such as, specific surface, surface charge density, exchange cation, and clay type.

# THE MEASUREMENT OF CONDUCTIVITY AND DIFFUSIVITY

The capillary conductivity and diffusivity vary markedly as a function of water content within a given soil and from soil to soil. Many of the features which characterize unsaturated flow are due to the extent of this variation. The fact that these quantities vary with water content makes them somewhat difficult to measure directly. Steady



Figure 2. Capillary conductivity of three soils plotted as a function of soil suction on a log-log scale.

state measurements of k can be made fairly readily within the tensiometer range by a method used by Moore (11). In this method a water table is maintained at the bottom of a column of soil and evaporation is allowed to take place at the top. When evaporation is steady, the suction gradient is measured by means of tensiometers in the column and the flux is given by the evaporation rate. A wide range of suction values can be obtained within columns about 100 cm long. At suctions above the tensiometer range, indirect methods are required. Richards and Moore (21) placed the soil samples in a pressure chamber and by applying appropriate pressures raised the pressure in the soil water to atmospheric so that manometers could be used to measure the potential differences. A similar procedure was employed by Corey (2), who measured simultaneous movement of water and air. Measurements of D or k in transient systems ordinarily require numerical or graphical solution of the flow equation, as was done by Bruce and Klute (1) and Richards and Weeks (22). For measurements in the field, it is most convenient to replace the flow equation by a finite difference equation. Such a procedure was used by Staple and Lehane (23).

In a method developed by this author (4) and extended by Miller and Elrick (10), the flow equation is solved for the removal of water from a soil in the pressure plate apparatus. The water content is changed by small increments so that the diffusivity can be assumed to be constant during this change. The diffusivity and the conductivity can then be calculated from the rate and quantity of outflow. Figure 2 shows conductivity data obtained in this manner plotted as a function of suction. The many-fold variation of conductivity with suction is readily noted. While the coarser textured soils have higher conductivities at the low suctions, their conductivities drop off much more rapidly with increasing suction and are generally less than for the finer textured soils at the higher suctions. For a number of soils studied, the conductivity can be related to the suction quite well by an expression of the form

$$k = \frac{a}{S^{n} + b}$$
(7)

where S is the suction, and a, b, and n are constants. The ratio a/b is equal to the saturated permeability. For most soils, n seems to lie between 1 and 4, al-

though for sands of fairly large uniform particles it may become considerably larger. In Figure 2 a value of n = 2 for the Chino clay and n = 3 for the Pachappa sandy loam approximates the data quite well.

Figure 3 shows the diffusivities for these same three soils plotted as a function of water content on a weight basis. The diffusivity varies over a wide range of values also but not quite to the extent that the conductivity does. In addition to the soils represented in Figure 3, the diffusivities of several other soils can be related to the moisture content by an exponential function

$$D = D_0 \exp \beta(\theta - \theta_0)$$
(8)

where  $D_0$  is the diffusivity when  $\theta = \theta_0$  and  $\beta$  is a constant.

If either k or D is known, the other can be calculated from the retention curve by means of Eq. 6.

## SOLUTION OF THE FLOW EQUATION

Analytical solutions of the flow equation cannot be hoped for except in a very few instances. Where Eq. 7 is applicable, it may be possible to integrate the flow equation for some steady state problems. Even if Eq. 7 is only very approximate, it may be possible to set limits on the flow process and obtain useful results. For the most part, particularly for transient problems, numerical methods of solution of the flow equation will be necessary. Such methods also offer the only real possibility of treating inhomogeneous and nonisothermal systems. Developments in the solution of the flow equation for transient problems have followed closely the treatment of nonlinear diffusion in other fields (3). Only a few generalizations will be considered.

Figure 3. Diffusivity of three soils plotted as a function of water content by weight on a semi-log plot.



#### Steady State Problems

For steady state conditions  $\partial \theta / \partial t = 0$ , and Eq. 5 becomes

$$-\nabla \cdot k\nabla S + \frac{\partial k}{\partial z} = 0 \tag{9}$$

This equation can be solved by replacing Eq. 9 by a finite difference equation and using a relaxation procedure. This Luthin and Day (9) did in a study of flow above a sloping water table. In treating two and three dimensional problems, it is sometimes helpful to define a new variable

$$U = \int_{S_0}^{S} k dS = \int_{\theta_0}^{\theta} D d\theta$$
(10)

where this integration may be performed numerically if necessary. Introduction of this variable into Eq. 9 gives

$$\nabla^2 U + \left(\frac{\partial k}{\partial U}\right) \frac{\partial U}{\partial z} = 0 \tag{11}$$

and the boundary conditions are also transformed in terms of U(x, y, z). When the gravitational term is negligible, Eq. 11 reduces to Laplace's equation for which solutions for the appropriate boundary conditions may be available. When the gravitational term is not negligible, Eq. 11 may be separated into three equations, one for each dimension, by the method of separation of variables.

For one dimensional flow, the problem is considerably simplified. Eq. 9 then becomes

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[ k \left( \frac{\mathrm{d}S}{\mathrm{d}z} - 1 \right) \right] = 0 \tag{12}$$

Integration of Eq. 12 once gives

$$k \left(\frac{dS}{dz} - 1\right) = q \tag{13}$$

where the constant of integration, q, represents the flux per unit area. Solving for z then gives

$$z = \int \frac{dS}{1 + q/k}$$
(14)

If k is known as a function of S this integration can be performed, numerically if necessary, and analytically for some cases (5).

### Nonsteady State Problems

Methods of solving the diffusion equation for variable diffusivity have been discussed by Crank (3) and will not be considered in any detail. In infinite or semi-infinite systems with certain boundary conditions, the Boltzmann transformation is useful. A new variable

$$y = x/2\sqrt{D_0 t}$$
(15)

reduces the partial differential equation to an ordinary differential equation which is more readily solved numerically. For uniform initial conditions and a constant boundary condition, Philip (12) has developed a rapid method of solution which he has recently extended to include the gravitational term (13). For finite systems, replacement of the partial differential equation by a finite difference equation as in the Crank-Nicolson method (3, p. 207) is most promising. The increased availability of high speed computing devices offers the greatest hope that solution of many of the unsaturated flow problems will become feasible. To date such devices have received only limited use in the solution of soil waterflow problems.

### SOLUTIONS OF SOME PARTICULAR FLOW PROBLEMS

Quantitative studies on unsaturated flow in soils have been directed mainly to investigations of the relative importance of various flow mechanisms and to testing the validity of Eq. 1. Of those few boundary problems for which solutions have been attempted, three will be mentioned here: (a) the entry of water into a soil at uniform initial water content, (b) the steady state evaporation of water from a soil in which there is a water table, and (c) the drying of a soil at uniform initial water content.

### Entry of Water into Soil

The infiltration of water into soil has been treated in considerable detail by Philip (14, 15, 16, 17, 18). In the early stages of infiltration, the gravitational term in the equation is negligible. Introduction of the Boltzmann transformation into the flow equation for a semi-infinite system gives the ordinary differential equation

$$-\frac{y}{2}\left(\frac{d\theta}{dy}\right) = \frac{d}{dy}\left(\frac{D}{D_0}\frac{d\theta}{dy}\right)$$
(16)

The water content,  $\theta$ , is now a function of y only. For a given  $\theta$ , say, that just back of the wetting front, x is proportional to  $\sqrt{t}$ . This explains the fact that the distance to the wetting front during infiltration is observed to increase as  $\sqrt{t}$ , or very nearly so. It can further be shown that the rate of entry of water into the soil is inversely proportional to  $\sqrt{t}$ . This relationship holds for any dependence of D on  $\theta$  for horizontal flow and the early stages of vertical flow if the initial water content is uniform. 'Using Philip's method, the author has obtained numerical solutions of Eq. 16 for exponential diffusivities (7). Figure 4 shows the water content distribution in horizontal samples of Chino clay for three different initial water contents compared with numerical solutions of the flow equation.



Figure 4. Water content distribution during horizontal flow of Chino clay for three different initial water contents. D<sub>s</sub> is the diffusivity at saturation and D<sub>o</sub> the diffusivity at the initial water content. The smooth curves represent theoretical solutions of the flow equation for an exponential diffusivity-water content relation.

The results are somewhat modified when the gravitational term is included and this term increases in importance with time. Philip (14) showed that for this case x can be represented as a power series in  $\sqrt{t}$  with the coefficients functions of ( $\theta$ )

$$x = \sum_{m=1}^{\infty} f_m(\theta) t^{m/2}$$
(17)

and likewise the cumulative infiltration can be represented by a similar power series

$$Q = \sum_{m=1}^{\infty} a_m(\theta) t^{m/2}$$
(18)

where the  $a_m$  are again functions of  $\theta$ , k, and D. For most soils, this series con-

verges rapidly and only the first two or



Figure 5. Relative rate of evaporation from a soil as a function of the suction at the soil surface for the case  $k = a/(S^3 + b)$  and with a water table 180 cm below the soil surface.

three terms are needed. Philip also relates the solution of the equation to several useful but more nearly empirical infiltration equations (17).

## Steady State Evaporation

The upward movement of water in a soil from a water table and its evaporation at the surface under steady state conditions can be described by solving Eq. 15. This has been done for several cases when Eq. 7 is obeyed and the constant n in that equation is an integer (5, 6). For values of n greater than one, the general features of the solutions are similar. Figure 5 shows the relative evaporation rate calculated for a soil for which n = 3 with a water table 180 cm below the soil surface. A low suction value at the soil surface occurs when the potential evaporation due to external conditions is relatively low and evaporation under these circumstances is controlled by the external conditions. However, as the potential evaporation, and therefore the suction at the soil surface, increases, the rate of evaporation from the soil rapidly increases until it approaches a limiting value. This limiting value is virtually attained by the time the suction at the soil surface reaches 1 or 2 bars. Thus, evaporation is limited by the potential external evaporation or the limiting rate of upward movement, whichever is the smaller.

The limiting rate of upward movement decreases with increasing depth to the water table and is given by

$$q_{\lim} = \frac{A_n a}{d^n}$$
(19)

where a is the constant in Eq. 7, d is the depth to the water table and  $A_n$  is a constant determined from the solution of the flow equation. The movement of water in the vapor form near the soil surface can be taken into account in a simple approximate manner (5). This has the effect of increasing  $A_n$  by a factor of probably not more than 10 to 15 percent.

The results indicate that the exact dependence of the conductivity on the suction is not critical, particularly at the higher suctions, and small deviations from Eq. 7 do not affect the results appreciably. If the soil profile is not homogeneous, as is probable, it can often be treated as a composite of several homogeneous layers and with the solutions for adjacent layers matched at their common boundary.

#### The Drying of Soils

The diffusion equation has been applied to the drying of clays and other porous media by a number of workers. Except for the study by van Vorst (24) the variation of the diffusivity with moisture content has not usually been introduced into the equations.

Certain analogies can be made with the case of steady state evaporation. In the early stages of the drying of a soil, the rate of evaporation is limited by the external conditions and if these are constant the evaporation rate is constant. In order to maintain this constant rate, the water content gradient through the soil increases with time and the water content at the boundary decreases. When the water content at this boundary reaches a value in equilibrium with the atmosphere, the constant drying rate can no longer be maintained and the drying rate then falls off. The length of time the rate is constant depends upon the magnitude of the rate and the properties of the soil. Under high evaporative conditions, this rate can be maintained for only a day or so by many soils. The maximum rate drying is achieved when the water content at the surface is brought immediately to the equilibrium value. In practice, this may be approached but can be achieved only by an infinitely high initial evaporation rate.

As in the case of water entry, application of the Boltzmann transformation to the equation for the drying of a semi-infinite soil sample in the absence of gravity, allows certain generalizations. It can be shown that the maximum accumulative evaporation should increase as  $\sqrt{t}$  if the initial water distribution is uniform. Crank (3, p. 256) points out that desorption processes can be described by a weighted mean diffusivity of the form

$$\overline{D} = \frac{1.85}{(\theta_i)^{1.85}} \int_0^{\theta_i} (\theta_i - \theta)^{.85} D(\theta) d\theta \qquad (21)$$

while, similarly, sorption processes can be described by a different weighted mean diffusivity



Figure 6. Theoretical relative water content distribution during drying of an infinite soil plotted as a function of the reduced variable  $x/2\sqrt{D_0 t}$ , for exponential diffusivities.  $\theta_i$  is the initial water content,  $\theta_0$  is the water content at the surface and  $D_i$  and  $D_0$  are the corresponding diffusivities. Gravity is neglected.

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$$\overline{D} = \frac{5}{3(\theta_i)^{5/3}} \int_0^{\theta_i} D(\theta) d\theta \qquad (22)$$

for a variety of diffusivity-concentration relations, including exponential. The higher diffusivities are weighted more heavily for the sorption than the desorption, so, for corresponding initial and final water contents, a soil will absorb water more rapidly than it will evaporate it. The greater the variation of diffusivity with water content the more marked this difference in rates. Philip (14) suggests that his method of solution of the flow equation can also be used to treat drying of soils when gravity is taken into account. A solution for this case has not yet been carried out, however, solutions have been obtained for the horizontal case for an exponential diffusivity (8). The solution for the semi-infinite case applies also to the early stages of evaporation from a finite soil.

Figure 6 shows the distribution of water in a semi-infinite soil sample with initial water content  $\theta_i$  and with the surface at x = 0 maintained at water content  $\theta_0$ .



Figure 7. The theoretical fractional loss of water from a finite soil sample plotted as a function of the reduced dimensionless variable  $\sqrt{D_0} t/L^2$  where L is the length of the sample. Each curve is for a different value of the ratio  $D_1/D_0$  when D is an exponential function of water content. Gravity is neglected.

Curves for several different ratios  $D_i/D_0$  are given, where  $D_i$  is the diffusivity at water content  $\theta_i$ , and  $D_0$  the diffusivity at water content  $\theta_0$ . Using a finite difference form of the flow equation, solutions have been obtained for the drying of a finite sample for an exponential diffusivity. Some of these solutions are shown in Figure 7 where the relative loss of water from the entire sample is plotted as a function of a convenient dimensionless ratio  $(D_0 t/L^2)^{1/2}$  where L is the length of the sample. Curves for several ratios of  $D_i/D_0$  are represented.

It is to be expected that for evaporation from a soil profile these curves will be somewhat modified by the inclusion of gravity, which tends to decrease the evaporation rate, and vapor movement, which tends to increase it. However, the general conclusions should be valid.

#### DISC USSION

In addition to helping understand unsaturated flow problems of practical importance, the flow equation and its solutions can be helpful in studying the role of various soil and water properties in their influence on water movement. The equation provides means to evaluate the diffusivity and the conductivity. Departures of real systems from theoretical solutions of the equation indicate the importance of temperature gradients, vapor movement, inhomogeneities, concentration gradients, and other effects which are usually otherwise difficult to study. Studies of the diffusivity and the conductivity as functions of water content, temperature, specific surface, particle size and arrangement, etc., should facilitate an understanding of the mechanisms involved in water movement. Although in practice it may seldom be feasible to attempt complete solutions of the flow equation, solutions for certain ideal cases can provide upper and lower limits to the rate of movement and a greater insight into flow processes.

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