Chapter 1

HYDRODYNAMIC APPROACHES

PART I

1.1 INTRODUCTION

In recent years, numerous mathematical theories of traffic flow applicable to long crowded roads have been developed. Although many of these theories involve a statistical approach, several are described in terms of fluid or hydrodynamic flows. The latter regard traffic as a compressible fluid having a certain density or concentration and a certain fluid velocity. Their analyses are based on a partial differential equation expressing the conservation of matter and an assumed empirical relation between the flow and the concentration. These analyses can be adjusted to fit flow-concentration curves of particular highways. The solution of the equation indicates that discontinuities in traffic flow are propagated in a manner similar to "shock waves" in the theory of compressible fluids. It is, therefore, the purpose of this chapter to discuss the application of fluid flow principles to the traffic stream.

1.2 FUNDAMENTAL CONCEPTS

Lighthill and Whitham prepared an outstanding paper on the theory of traffic flow in which they discussed the behavior of shock waves in the traffic stream and developed a theory of the propagation of changes in traffic distribution. Part I is an elementary approach to this theory.

Consider the movement of two distinct concentrations of traffic k_1 and k_2 along a straight highway (Fig. 1.1). The two concentrations k_1 and k_2 are separated by the vertical line S, which has a velocity of c. This velocity is considered positive if the line moves in the direction of positive x as depicted by the arrow. With the following notations:

u_1 = Mean speed of vehicles in region A;
u_2 = Mean speed of vehicles in region B;
$U_{r1} = (u_1 - c)$ = Relative speed of vehicles in region A to the moving line S; and
$U_{r2} = (u_2 - c)$ = Relative speed of vehicles in region B to the moving line S,

it is evident that in time t the number of vehicles N crossing the dividing line S is

$$N = U_{r1}\, k_1\, t = U_{r2}\, k_2\, t \qquad (1.1)$$

or

$$(u_1 - c)\, k_1 = (u_2 - c)\, k_2 \qquad (1.2)$$

This equation is a statement of the conservation of matter applied to the vehicles that cross the line S and may be written in the form

$$u_1\, k_1 - u_2\, k_2 = c\,(k_1 - k_2) \qquad (1.3)$$

If the rate of traffic flow in region A is q_1, and the rate of traffic flow in region B is q_2,

$$q_1 = k_1\, u_1 \qquad (1.4)$$

and

$$q_2 = k_2\, u_2 \qquad (1.5)$$

These relations follow from the definition of the quantities involved. In terms of the rates of flow q_1 and q_2, Eq. 1.3 becomes

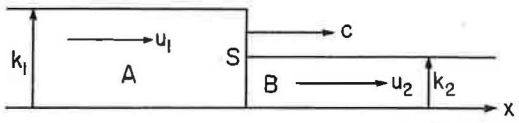

Figure 1.1. Movement of two concentrations.

$$c = (q_2 - q_1)/(k_2 - k_1) \qquad (1.6)$$

If the rates of flow and the concentrations are nearly equal,

$$(q-q_1)=\Delta q\ ,\ (k-k_1)=\Delta k \qquad (1.7)$$

and Eq. 1.6 becomes

$$c=\Delta q/\Delta k=dq/dk \qquad (1.8)$$

which is the equation for the velocity c with which small *disturbances* in the traffic stream are propagated.

In the general case in which the differences in the concentrations on the two sides of the moving line S are not infinitesimally small, Eq. 1.3 may be written in the form

$$c=(u_1 k_1-u_2 k_2)/(k_1-k_2) \qquad (1.9)$$

So far, the elementary analysis has not considered any relation between the mean velocities u_1 and u_2 and the concentrations k_1 and k_2. Greenshields (*1*) found in his study of traffic capacity that

$$u_1=\bar{u}_s(1-\eta_1) \text{ and } u_2=\bar{u}_s(1-\eta_2) \qquad (1.10)$$

in which $\bar{u}_s$ is the space-mean speed of the traffic stream, and η_1 and η_2 are the normalized concentrations on both sides of the boundary line S. Substituting these values in Eq. 1.9 gives a wave speed of

$$c=\frac{[k_1\,\bar{u}_s\,(1-\eta_1)-k_2\,\bar{u}\,(1-\eta_2)]}{(k_1-k_2)} \qquad (1.11)$$

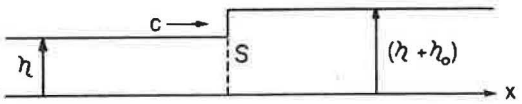

Figure 1.2. Small discontinuity in concentration.

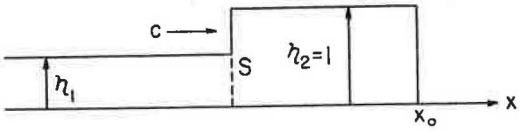

Figure 1.3. Shock wave caused by stopping.

The normalized concentrations η_1 and η_2 are given by

$$\eta_1=k_1/k_j,\ \eta_2=k_2/k_j \qquad (1.12)$$

in which k_j, the jam concentration, is the maximum concentration of vehicles when jammed at a stop. Both k_1 and k_2 may be eliminated from Eq. 1.11, the resulting wave speed being

$$c=\bar{u}_s\,[1-(\eta_1+\eta_2)] \qquad (1.13)$$

which gives the velocity of the line S in terms of the normalized concentrations on either side of the moving discontinuity.

1.2.1 The Case of Nearly Equal Concentrations

If the normalized concentrations η_1 and η_2 on both sides of the boundary line S are nearly equal, the situation shown in Figure 1.2 exists. The normalized concentration to the left of S is η, whereas the normalized concentration to the right of S is $(\eta+\eta_0)$, where $\eta+\eta_0\leq 1$. In this case, let

$$\eta_1=\eta\ ,\ \eta_2=[\eta+\eta_0] \qquad (1.14)$$

and

$$[1-(\eta_1+\eta_2)]=[1-(2\eta+\eta_0)]=[1-2\eta] \qquad (1.15)$$

in which η_0 is neglected. If Eq. 1.13 is substituted in Eq. 1.15, the wave of discontinuity is propagated with a velocity of

$$c=\bar{u}_s[1-2\eta] \qquad (1.16)$$

This is the equation for the propagation of shock waves obtained by Lighthill and Whitham by a more elaborate analysis.

1.2.2 Waves of Stopping

Consider a line of traffic moving with a normalized concentration η_1 and a mean vehicle velocity of

$$u_1=\bar{u}_s[1-\eta_1] \qquad (1.17)$$

At a position $x=x_0$ on the highway, a traffic signal causes the traffic to halt, and the stream immediately assumes a saturated normalized concentration of $\eta_2=1$, as shown in Figure 1.3. To the left of the line S, the traffic is still moving with a mean velocity

given by Eq. 1.17 at the original concentration of η_1. Under these conditions, the shock wave velocity is given by substituting $\eta_1=\eta_1$ and $\eta_2=1$ in Eq. 1.13 to give

$$c=\bar{u}_s[1-(\eta_1+1)]=-\bar{u}_s\eta_1 \quad (1.18)$$

which indicates that the shock wave of stopping travels backward with a velocity of $\bar{u}_s\eta_1$. If the signal at $x=x_0$ turns red at $t=0$, then in time t later, a line of cars of length $\bar{u}_s\,\eta_1 t$ will be stopped behind x_0.

1.2.3 Waves of Starting

In order to discuss the nature of the shock wave produced by the starting of a line of vehicles, assume that at $t=0$ a line of vehicles has accumulated behind a signal located at $x=x_0$. Because this line of vehicles is standing still, it has a saturated concentration of $\eta_1=1$, as shown in Figure 1.4. Assume that at $t=0$ the signal at $x=x_0$ turns green and permits vehicles to move forward with a velocity of u_2. Because $u_2=\bar{u}_s[1-\eta_2]$ there exists a concentration of

$$\eta_2=[1-(u_2/\bar{u}_s)] \quad (1.19)$$

Therefore, a shock wave of starting forms as soon as the line of vehicles begins to move. The velocity of this shock wave is obtained by substituting $\eta_1=1$ and $\eta_2=\eta_2$ in Eq. 1.13, thus

$$c=\bar{u}_s[1-(1+\eta_2)]=-\bar{u}_s\,\eta_2=-(\bar{u}_s-u_2) \quad (1.20)$$

Therefore, the shock wave of starting travels backward from x_0 with a velocity of $(\bar{u}_s-u_2)$. Because the starting velocity is small, it is seen that the shock wave of starting travels backward with a velocity essentially equal to $-\bar{u}_s$.

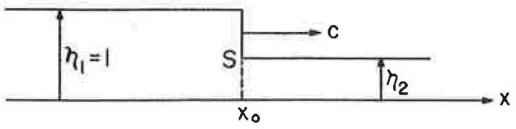

Figure 1.4. Shock wave caused by starting.

1.3 COMPARISON OF LIGHTHILL-WHITHAM AND RICHARDS THEORIES

Richards (*2*) prepared a paper on the theory of traffic shock waves, covering the same material as Lighthill and Whitham, at about the same time and without knowledge of their work.

Essentially these two theories are identical. Lighthill and Whitham center their attention on the discontinuities in the rate of flow q, whereas Richards centers his attention on the discontinuities in the concentration k, which he calls the density function D. In both theories the fundamental equation is the one that expresses the conservation of matter. However, because Lighthill and Whitham do not restrict themselves to any definite flow-concentration curve, their analysis is somewhat more general than that of Richards.

Richards incorporates in his basic equations the straightline relation $u=\bar{u}_s\,(1-\eta)$ for the mean velocity of the vehicles. Therefore, the conclusions reached by Richards are limited to situations in which this law of velocity and concentration hold. If this hypothesis is incorporated into the Lighthill-Whitham theory, their theory is identical with that of Richards. The difference between the two theories is then seen to be only one of notation and graphical interpretation.