Abridgment

Probability of Sliding of Soil Masses

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The employment of logarithmic spiral failure surfaces in slope stability analysis was dictated by the need to bring analytical models closer to the configuration of actual failures (1). Initially, such surfaces were associated with unity safety factor and thus were used only for computations of the critical heights of slopes (2). The first attempt to adapt the log-spiral against sliding was made by Fröhlich (3). The analytical expression used was the following:

$$r = r_0 e^{-\theta t} \quad (1)$$

where

r_o = the initial radial vector,
$t(=\tan\phi)$ = the soil strength parameter, and
θ = the angle between r_o and r (Figure 1).

The factor of safety (F_s) with respect to sliding of a soil mass is equal to (3)

$$F_s = M_R/M_S \quad (2)$$

where M_R = the moment around an axis of all resisting forces acting on the sliding mass along the sliding surface, and M_S = the moment of all forces driving the soil mass towards sliding.

After moments M_R and M_S are expressed analytically, Equation 2 becomes Equation 3

$$F_s = M_R/M_S = 1 + \{(c/S)[(r_0^2 - r_H^2)/2t] - a_0\} \div [a_0 + r_H e^{\mu\theta} H^t \times t \times \sin(\mu\theta_H + \delta)] \quad (3)$$

where

r_o, r_H = radial vectors (Figure 1a),
S = the resultant of all driving forces acting on the sliding mass (Figure 1b),
a_o = the distance between the center of the spiral and S (Figure 1a),
δ = the angle between r_H and the normal to the direction of S,
θ_H = the angle between r_o and r_H,
μ = a parameter that determines the center of rotation and receives values between zero and infinity, and
$c, t(=\tan\phi)$ = the soil strength parameters.

In the above equation, all variables are treated as single-valued quantities. However, in soils, values of material parameters exhibit a considerable variation (4). The same is true (1) for the geometric factors in Equation 3 (i.e., angles θ_H and δ; distances r_o, r_H, and a_o; and quantity μ). Thus, a more reliable approach to the measure of safety of a soil slope must take into account uncertainties such as material parameters and the shape and location of the failure surface.

PROBABILITY OF SLIDING

In recent years, the use of probability theory and statistical analysis has provided an alternative to the determination of the factor of safety. In its classical formulation, sliding of a soil slope is assumed to occur when the calculated moment of all resisting forces (M_R) becomes smaller than that of the driving forces toward failure (M_S); that is, Sliding = $[M_R < M_S]$. The probability of failure in sliding is then defined as $p_f = P[M_R < M_S] = P[M_R/M_S < 1]$, where P[ ] indicates the probability that the driving forces exceed available strength. In Equation 3 the expression $M_R/M_S = F_s < 1$ is identical to $c/S[(r_o^2 - r_H^2)/2t] - a_o < 0$, i.e., the numerator of the second term of the right side of Equation 3 becomes negative. Therefore, the expression for p_f becomes

$$p_f = P\{(c/S)[(r_0^2 - r_H^2)/2t] - a_0 < 0\} \text{ or}$$
$$p_f = P[c(r_0^2 - r_H^2)/2t < Sa_0] \quad (4)$$

The quantity $(r_o^2 - r_H^2)/2t$, in Equation 4, is equal to twice the area (A) of the region OBMAO, shown in Figure 1a, or

$$(r_0^2 - r_H^2)/2t = 2A \quad (5)$$

and the quantity Sa_o gives the moment, say M_o, of the driving forces around the center of the sliding surface; i.e., $M_o = Sa_o$. In the case where the value of the factor of safety is equal to unity, or (2c) A = M_o, the moment M_o varies in proportion to the area A, the coefficient of proportionality being the double value of the c strength parameter. If this product is denoted by M, i.e., if

$$M = 2cA = c(r_0^2 - r_H^2)/2t \quad (6)$$

and the expressions for M and M_o are introduced into Equation 4,

$$P_f = P[M < M_0] \quad (7)$$

The uncertainty of the value of M reflects the uncertainties of the strength parameters c and t and, also, of the location of the center 0 of the sliding surface (the latter is determined by the geometric factors h_o and θ_o). From Equation 6 it can be seen that the value of M (random variable) ranges between zero (lower limit) and infinity (upper limit). As was the case with other random variables (1), it can be assumed that M follows a log-normal distribution. Under this assumption, the probability density function of M is (5)

$$f(M) = 1/(\sqrt{2\pi}\,\sigma_M M) \exp\{-½[(\ln M - \overline{M})/\sigma_M]^2\} \quad (8)$$

where $0 \le M < \infty$ and $\overline{M}$ and σ_M denote the mean value and standard deviation of M, respectively.

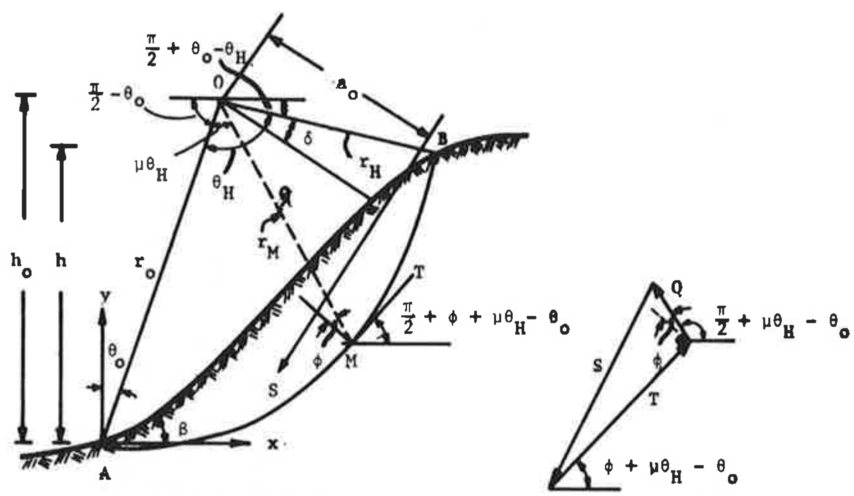
Combining Equations 7 and 8, the following expression for the probability p_f of sliding of the soil mass is then derived:

$$p_f = \int_0^{M_0} f(M)dM = F(M_0) \quad (9)$$

where $F(M_o)$ = the cumulative log-normal distribution evaluated at M_o.

As M is taken to be log-normally distributed, the

Figure 1. Logarithmic-spiral sliding surface and polygon of forces.



a. Log-spiral Mode of Sliding

b. Polygon of Forces T,Q,S

variable x, which is equal to lnM, is normally distributed. If the coefficient of variation and mean value of x are denoted by S_x and μ_x, respectively, then

$$S_x = [\ln(V_M^2 + 1)]^{1/2} \tag{10a}$$

and

$$\mu_x = \ln(\overline{M}) - S_x^2/2 \tag{10b}$$

Introducing the normalized variable z, defined as $z = (\ln M - \mu_x)/S_x$, the expression for the probability of sliding p_f becomes:

$$p_f = P[M < M_0] = P[z < z_0] = \Psi(z_0) \tag{11}$$

where z_o = the value of z evaluated at $M = M_o$, and $\Psi(\,)$ = the well-tabulated cumulative standard normal distribution.

STATISTICAL VALUES OF MOMENT (M)

Equations 10 and 11 can be solved provided that the mean value ($\overline{M}$) and standard deviation (σ_M) of M are determined. For a given surface of sliding (i.e., $\frac{1}{2}(r_o^2 - r_H^2)$ = constant = b), Equation 6 suggests that the value of moment M depends on the strength parameters c and t and the geometrical constants r_o and r_H. Equation 6 can be reduced to

$$M = b\,c/t \tag{12}$$

where $b = \frac{1}{2}(r_o^2 - r_H^2)$. If c and t are independent random variables, the mean value $\overline{M}$ and variance Var(M) of M can be found by means of a Taylor series expansion of the function M(c, t) around the point $M(\overline{c}, \overline{t})$, where $\overline{c}$ and $\overline{t}$ denote the mean values of c and t, respectively (6); i.e.,

$$\overline{M} = M(\overline{c}, \overline{t}) + \tfrac{1}{2}\{[(\partial^2\overline{M})/\partial c^2]\ \mathrm{Var}(c) + [(\partial^2\overline{M})/\partial t^2]\ \mathrm{Var}(t)\} \tag{13a}$$

$$\mathrm{Var}(M) = (\partial\overline{M}/\partial c)^2\ \mathrm{Var}(c) + (\partial\overline{M}/\partial t)^2\ \mathrm{Var}(t) \tag{13b}$$

where Var(c), Var(t) are the variances of c and t, respectively, and the derivatives are evaluated at the mean values of the variates.

From Equation 12 one has

$$\partial M/\partial c = b/t,\ \partial^2 M/\partial c^2 = 0,\ \partial M/\partial t = -bc/t^2,\ \partial^2 M/\partial t^2 = 2bc/t^3 \tag{14}$$

Combining Equations 13 and 14,

$$\overline{M} = b\ [\overline{c}\overline{t}^2 + \overline{c}\ \mathrm{Var}(t)]/\overline{t}^3 \tag{15a}$$

and

$$\mathrm{Var}(M) = b^2\ [\overline{t}^2\,\mathrm{Var}(c) + \overline{c}^2\,\mathrm{Var}(t)]/\overline{t}^4 \tag{15b}$$

The coefficient of variation V_M of M can be determined from Equations 15 as

$$V_M = \sigma_M/\overline{M} = \{\overline{t}/[\overline{c}\overline{t}^2 + \overline{c}\mathrm{Var}(t)]\}\ [t^2\,\mathrm{Var}(c) + c^2\,\mathrm{Var}(t)]^{1/2} \tag{16}$$

where V_M is independent of the constant b.

In Figure 2, the mean value of the quantity $\overline{M}/b$ is plotted versus the mean value ($\overline{t}$) of the strength parameter t for various values of $\overline{c}$. Studies of the variability of the soil strength parameters t and c have indicated (4) that their coefficients of variation V_t and V_c approach values of 15 and 70 percent, respectively. These same values for V_t and V_c have, therefore, been adopted in this paper.

EXAMPLE

The slope shown in Figure 3 has a height h = 9.75 m (32 ft) and an angle $\beta = 40°$. The mean values and coefficients of variation of the strength parameters of the soil, determined from conventional triaxial tests, are also shown in the figure. The moist unit weight of the soil is assumed to be 1762 kg/m³ (110 lb/ft³). The probability of failure, or the reliability of this slope, is to be determined.

Three random factors in Equation 1 reflect the uncertainty of (a) the location of the center of the rupture surface, (b) the point of initiation of the rupture surface, and (c) the value of the t-strength parameter of the soil material. A procedure necessary to generate such surfaces was presented previously (1). The assumptions involved are that the failure surface begins at the toe of the slope and the coordinates θ_o and h_o of the center of the log-spiral (Figure 1) follow a beta distribution. In Figure 3, the mean failure surface can be found through an application of the Monte Carlo simulation technique

Figure 2. Influence of the statistical values of strength parameters on the mean value of the moment (M).

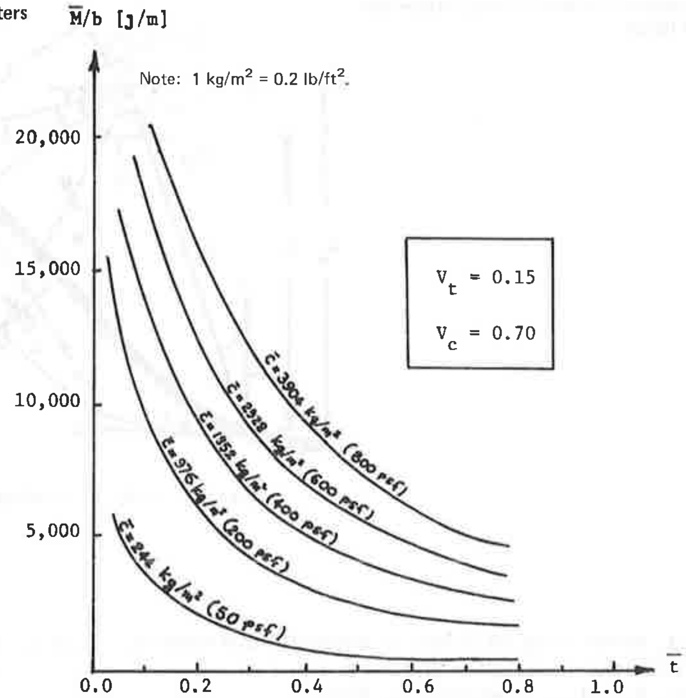
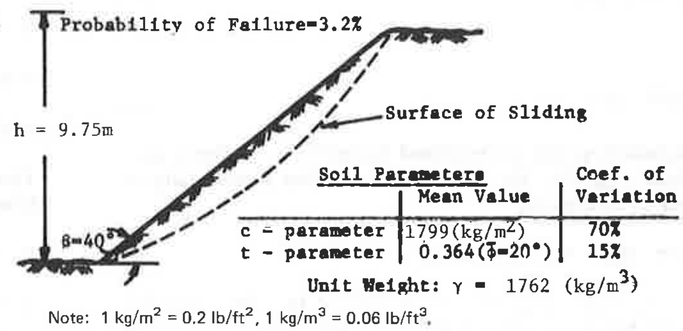


Figure 3. Slope section and soil parameters used for the case study.



(1). The center of the log spiral has coordinates h_o = 22.81 m (74.83 ft) and $\theta_o = 0°$ (Figure 1). The quantities r_H and θ_H are thus equal to 17.42 m (57.14 ft) and 43°, respectively.

The driving moment (M_o) due to the weight of the sliding soil mass is equal to $M_o = Wa_o$ (3), where a_o is shown in Figure 1. Area A (determined by Equation 5) and distance a_o (measured graphically in Figure 3) are equal to 42.67 m^2/m (140 ft^2/ft) and 6.1 m (20 ft), respectively. Therefore, $M_o = \gamma \cdot A \cdot a_o$ = (1762) (42.67) (6.1) = 458 626 J/m (154 ft-tons/ft).

From Figure 2, for $\bar{t}$ = 0.364 and $\bar{c}$ = 1797.3 kg/m^2 (368.3 lb/ft^2) we find $\overline{M}/b \approx$ 5053 kg/m^2 (1035 lb/ft^2), where $b = (r_o^2 - r_H^2)/2 = [(22.81)^2 - (17.42)^2]/2$ = 103.15 m^2 (1167.27 ft^2).

Therefore, $\overline{M}$ = (5053) (103.15) = 521 234 J/m (604.1 ft-tons/ft).

The coefficient of variation M (V_M) is found from Equation 16 to be equal to 70 percent.

For $\overline{M}$ = 521.234 J/m^2 (604.1 ft-tons/ft) and V_M = 70 percent, Equation 10 yields S_x = 0.6315 and μ_x = 12.96. The probability of sliding of the slope can now be determined from Equation 11 as follows:

$$p_f = P[z < z_0] = \Psi(-1.8485) \approx 3.2\% \quad (17)$$

DISCUSSION AND CONCLUSIONS

Statistical analysis and probability theory can be used as alternatives to conventional (deterministic) methods for evaluation of slope stability. In this paper, the reliability of a soil slope against sliding was evaluated from its probability of failure. This was defined as the probability that the resisting moment M_R was exceeded by the driving moment M_S. Sliding was assumed to occur along a log-spiral path. This assumption is consistent with results obtained through stochastic modeling of the propagation of failure surfaces (7).

As the variation of the unit weight (γ) of the soil is relatively small, γ was assumed to be constant. Thus, moment M was expressed as a function of only two random variables: strength parameters c and t. The variability of c and t in the expression for the moment M was investigated by means of a Taylor series expansion of M(c, t) around the point $M(\bar{c}, \bar{t})$. It should be noted that this method gives only approximate values for the mean and variance of M. If greater accuracy is required, a more precise procedure, possibly a simulation approach, should be employed.

In the illustrative example it was found that the probability of failure of the slope was approximately 3.2 percent or, out of 100 identical slopes, on the average, 3.2

would fail. The reliability of this slope is then said to be equal to 96.8 percent.

Based on the results of this study, it is concluded that

1. The probabilistic model developed here can be used to find a value of the probability of failure (or, the reliability) of a soil slope. This depends on the slope geometry and on the statistical values of the soil parameters.

2. The method can be applied to either deep or shallow failures. The kind of failure is reflected in the probability density functions of the coordinates of the center of the sliding surface.

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Soil-Culvert Interaction Method for Design of Metal Culverts

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A simple and rational method for the design of metal culverts, the soil-culvert interaction method, is described and compared to currently used design procedures. The principal advantage of the soil-culvert interaction method over those previously developed is that it provides a logical procedure for determining minimum required depth of cover, by consideration of the bending moments caused by live loads. Previously, minimum depths of cover have been determined empirically, using field experience. Values of minimum cover and maximum fill height determined using the soil-culvert interaction method are compared with values from published fill-height tables. The comparisons show that the soil-culvert interaction method gives values that are in good agreement with design experience for a wide range of corrugations and culvert diameters.

A simple method for design of metal culvert structures has been developed to provide rational procedures for designing culverts with deep or shallow cover. Design for deep cover is based on consideration of ring compression forces. Design for shallow cover is based on consideration of both ring compression forces and bending moments. The method, the soil-culvert interaction (SCI) method, is applicable to circular pipes, pipe arches, and arches constructed of corrugated steel or aluminum. It may be applied to structures having stiffening ribs that are curved to conform to the shape of the culvert barrel and attached to the barrel at frequent intervals. However, it is not applicable to soil bridge structures, which use straight ribs, fin plates, and sometimes strut to stiffen the upper part of the structure. The SCI method has been found to give values of maximum and minimum cover that are in good agreement with design experience as reflected in published fill-height tables and with the observed behavior of culverts in the field.

BASIS FOR SCI METHOD

The SCI design procedure is based on the results of finite element analyses, which modeled both the culvert structure and the surrounding backfill. Detailed results of the analyses and comparisons with field measurements were described by Duncan (1). Similar analyses were performed by Allgood and Takahashi (2), Abel and others (3), and Katona and others (4). The analyses on which the SCI method is based simulated the placement of backfill around and over the structure, and subsequent application of live loads on the surface of the backfill. Nonlinear and stress-dependent stress-strain relationships for the backfill soils were employed in the analyses. The results of these analyses were used to derive coefficients for ring compression forces and bending moments for design.

STEPS IN SCI DESIGN PROCEDURE

1. Calculate the rise/span ratio (R/S). The definitions of rise and span as used in this procedure are shown in Figure 1.

2. Calculate the maximum ring compression force

$$P = K_{p1}\,\gamma S^2 + K_{p2}\,\gamma HS + K_{p3}\,LL \qquad (1)$$

where

P = ring compression force (kN/m);
K_{p1} = ring compression coefficient or backfill, from Figure 2 (dimensionless);