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Optimal Bus Scheduling on a Single Route

YOSEF SHEFFI AND MORIHISA SUGIYAMA

This paper develops a method for scheduling bus runs on a single route so as to minimize the total waiting time for patrons. The demand for bus travel is assumed to be time-dependent with a given origin-destination pattern. The problem is formulated as a mathematical program subject to bus capacity and possible other (e.g., service standards) constraints. A dynamic programming procedure is suggested for the solution of this program. Finally, some issues associated with optimization of the schedule under stochastic demand are explored as well.

The conventional wisdom in setting up a schedule for a bus route typically involves supplying enough capacity at the maximum load point, subject to some service standards. This approach is useful when the period for which the schedule (timetable) is set is one in which the arrival rate of patrons is constant, and when this period is long in comparison with the bus roundtrip time on the route. When these conditions are not met (for example, the maximum load point may not be stationary or the peak period may be shorter than a bus trip), the schedule may not be optimal in the sense that unnecessary crowding may exist on some buses and slack capacity may exist on others. Better scheduling may bring about a higher level of service and increased productivity.

The problem referred to in this paper is that of preparing a desirable schedule for a given route, not of scheduling the actual buses to the runs. In other words, bus availability and interlining considerations are not taken into account. The inputs to the schedule preparation problem discussed here are the route geometry (including stops and speeds between stops), the total number of bus runs to be operated, and the desired trip rates (i.e., the demand pattern). The output is the route's schedule.

The objective of the schedule preparation is to find the timetable that would give passengers the maximum level of service for a given level of resources. The level of service is expressed in terms

of total waiting time that should be minimized, given a fixed number of bus runs.

The practice in the industry, which is also reflected in the literature, is to assume that the demand rate is constant throughout a given design period (e.g., the morning peak). The problem is then only to find the optimal frequency with which to operate the route. Once this frequency is determined, runs are evenly spaced to minimize waiting time. In reality, however, demand rates are not constant but vary over time even within design periods and, in fact, many bus schedules are not evenly spaced for any period. The focus of this paper is on the optimal schedule given a time-dependent demand pattern. The first four sections of this paper consider the demand rates to be deterministic (even though the rates change randomly from day to day). This is due to the planning and optimization perspective of the approach, which focuses on the mean service rates and run times rather than the variability in these measures. The last section attempts to consider some stochastic effects.

A schedule optimization for the simple case of one destination and where the stop dwell time is independent of the number of boardings is analyzed by Newell (1). Newell's approach is used as a starting point to the cases analyzed in this paper. Similar cases, with the inclusion of the bus round-trip constraint, were analyzed by Salzborn (2) and Hurdle (3,4).

The effect of boarding time on the scheduling problem was briefly discussed by Friedman (5) and extensively by the many researchers who looked at the bus bunching phenomenon [e.g., Osuna and Newell (6), Barnett and Kleitman (7), Newell (8), Chapman and Michel (9), and Jordan and Turnquist (10)].

The final section introduces some stochastic considerations and concludes that such modeling should actually be attempted in the context of real time control rather than in the context of scheduling.

This paper does not include numerical examples of the methods shown for reasons of brevity. The interested reader can find complete numerical examples with detailed solutions to all the problems discussed here in Sugiyama (11).

SIMPLE SCHEDULING PROBLEM

This section analyzes a bus route with multiple boardings and one destination point (the final stop). The stops are numbered consecutively $i = 0,1,2...,m$, and passengers are assumed to board at stops 1 through $m-1$ and are all destined for m (the starting point, 0, represents a garage or layover point, which does not have to be a part of the route). In this problem it is further assumed that the bus speed is constant where $\Delta t_i, i = 1,2...,m$ represents the bus travel times between stop $(i-1)$ and stop i, and the dwell times are assumed independent of the number of boardings.

The input to this problem includes the demand at each stop expressed in terms of the cumulative arrivals $[F_i(t)]$ during the design period; the number of bus runs available during the design period (T); and the interstop travel times (Δt_i). Following Newell (1), we define the shifted cumulative demand at point 0 as

$$F(t) = \sum_{i=1}^{m-1} F_i\left(t + \sum_{k=1}^{i} \Delta t_k\right) \tag{1}$$

Assume n buses are dispatched from 0 at times $t_j, j = 1,2,...,n$, as shown in Figure 1. Obviously the total waiting time (w) is the shaded area between $F(t)$ and the step function created by $F(t_j)$ for $t_j \leq t \leq t_{j+1}$.

In other words, the total wait time (w) is given by

$$w = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [F(t) - F(t_j)]\, dt \tag{2}$$

where $F(t_0) = 0$ and $F(t_n)$ is the total number of boardings along the route ($t_n = T$, the end of the design period).

Following Newell we observe that in order to minimize w, every passenger has to be picked up by the first bus that comes along after his or her arrival at the stop. This, in fact, can be thought of as a necessary condition for a minimum. This observation simplifies any solution procedures significantly because it reduces the feasible space, as shown below.

Newell presents the analytical solution to the minimization of w with no capacity constraints. In our case we are interested in minimizing w subject to the following constraints:

$$F(t_j) - F(t_{j-1}) \leq C \quad \text{for } j = 1, 2, \ldots, n \tag{3a}$$

$$F(t_0) = 0 \tag{3b}$$

and

$$0 < t_1 < t_2 \ldots < t_n = T \tag{3c}$$

where C is the capacity of a bus. The fixed capacity (C) can be readily replaced with C_j where each bus has a different capacity without affecting any of the solution procedures.

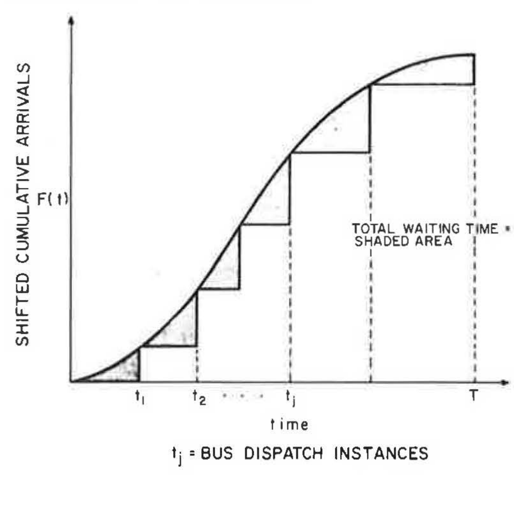
This problem can be solved by a simple dynamic programming (DP) procedure that uses the abovementioned necessary condition, linking the load on each bus with its dispatch time.

The DP procedure can be formulated in several ways for this program. For example, since the dispatch time of the last bus is fixed at $t_n = T$, the procedure can work backwards in time by using the recursive relation:

$$r_j(t_j) = \underset{t_j}{\text{MIN}} \left\{ r_{j+1}(t_{j+1}) + \int_{t_j}^{t_{j+1}} [F(t) - F(t_j)]\, dt \right\} \quad \text{for } j = n-1, n-2, \ldots 1 \tag{4}$$

where $r_j(t_j)$ is the total wait time for all passengers carried by the last $(n-j)$ buses where these buses have been dispatched optimally. The

Figure 1. Shifted cumulative arrivals.



stages of the system thus correspond to the bus dispatch decisions and the states to the dispatch times. The state dynamics are obvious in this formulation as the solution to each stage is the state variable itself. A forward recursion similar to Equation 4 can be formulated also since $F(t_0) = 0$. In any event the recursive relations are subject to the constraint set in Equation 3.

In order to execute the DP procedure, the time line can be divided into discrete parts, say whole minutes. Such a procedure would ignore the integer nature of the number of passengers who use each bus [this discrete nature of the problem is masked by the use of the continuous approximation $F(t)$ to the arrival pattern]. In the case where the number of patrons is small in comparison with the length of the design period and this approximation is inaccurate, the DP procedure can be formulated in terms of loads rather than dispatch times, making the load on the jth bus the state variable. (Alternatively, the time state space, in the original formulation, can be searched in a manner that would ensure the integer nature of the load.) Note, however, that the load and $F(t)$ correspond to the average conditions and therefore the number of passengers who use a particular bus does not have to be an integer on the average.

A simpler solution to the optimal dispatch problem can be obtained if the objective function is to equalize the load factors among buses rather than to minimize the waiting time. In such a case the optimal load on each bus is $s^* = (1/n)\ F(T)$ for all i, assuming that $s^* \leq C$. The schedule in this case can be found graphically or by the simple recursion:

$$t_{i-1} = F^{-1}\,[F(t_i) - s^*] \qquad (5)$$

This procedure will not, however, yield the minimum total wait. The minimum wait time schedule, in general, will be such that buses dispatched at a period where the slope of $F(t)$ is high should pick up more passengers than buses dispatched when $F(t)$ is relatively flat. Under the even load policy all buses will be equally loaded and therefore the waiting time will not, in general, be minimized.

SCHEDULING WITH MULTIPLE ORIGINS AND DESTINATIONS

The problem discussed in this section is identical to the optimal scheduling problem, but we assume that patrons can board and alight at any stop along the route. Thus, let P_{ij} represent the fraction of patrons who board at station i who alight at station j. P_{ij} is assumed to be constant over time (during the design period), and

$$\sum_{j=i+1}^{m} P_{ij} = 1 \qquad \text{for } i = 1, 2, 3, \ldots m - 1 \qquad (6)$$

The problem of finding the schedule that minimizes the total travel time is more complicated now because the capacity constraint cannot be formulated in a straightforward manner as in Equation 3a. In order to formulate this constraint let $G_i(t)$ denote the cumulative alightings at stop i by time t, in other words:

$$G_1(t) \equiv 0 \qquad (7a)$$

$$G_2(t) = F_1(t - \Delta t_2)P_{12} \qquad (7b)$$

$$G_3(t) = F_1(t - \Delta t_2 - \Delta t_3)P_{13} + F_2(t - \Delta t_3)P_{23} \qquad (7c)$$

$$\vdots$$

$$G_i(t) = \sum_{l=1}^{i-1} F_l\left(t - \sum_{k=l+1}^{i} \Delta t_k\right)\cdot P_{li} \qquad i = 2, 3, \ldots, m \qquad (7d)$$

where Δt_i is the bus travel time between stop (i-1) and i and $F_i(t)$ is the cumulative number of arrivals at the ith stop. In many cases $G_i(t)$ can be obtained directly from the measurements (it is easier to measure than P_{ij}) and P_{ij} need not be used [$G_i(t)$ will be used directly as input to the analysis].

Let $N_i(t)$ represent the cumulative number of passengers carried out of the ith station (imagine that we look at the system only at bus departure times). The functions $N_i(t)$ are obtained as follows:

$$N_1(t) = F_1(t)$$
$$N_2(t) = N_1(t - \Delta t_2) + F_2(t) - G_2(t)$$
$$N_3(t) = N_2(t - \Delta t_3) + F_3(t) - G_3(t)$$
$$\vdots$$
$$N_{m-1}(t) = N_{m-2}(t - \Delta t_{m-1}) + F_{m-1}(t) - G_{m-1}(t).$$

Therefore,

$$N_1(t) = F_1(t)$$
$$N_2(t) = F_1(t - \Delta t_2) + F_2(t) - G_2(t)$$
$$\vdots$$
$$N_i(t) = \sum_{k=1}^{i} \left[F_k\left(t - \sum_{\ell=k+1}^{i} \Delta t_\ell\right) - G_k\left(t - \sum_{\ell=k+1}^{i} \Delta t_\ell\right)\right].$$

where $\sum_{\ell=k+1}^{i} \Delta t_\ell \equiv 0$ for $k+1 > 1$ and $F_m(t) = G_1(t) = 0$.

Since the objective function in the problem is expressed in terms of shifted demands, let us shift $N_i(t)$ to the origin as well. Let $N_i'(t)$ denote the shifted cumulative number of passengers on link (i, i+1). It can be calculated as (see Equation 1)

$$N_i'(t) = N_i\left(t + \sum_{k=1}^{i} \Delta t_k\right) \qquad (8)$$

and, substituting the last expression for $N_i(t)$, we get

$$N_i'(t) = \sum_{k=1}^{i} \left[F_k\left(t + \sum_{l=1}^{k} \Delta t_l\right) - G_k\left(t + \sum_{l=1}^{k} \Delta t_l\right)\right]$$
$$\text{for } i = 1, 2, 3, \ldots m - 1 \qquad (9)$$

The minimization problem now is

$$\min w = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [F(t) - F(t_{j-1})]\, dt \qquad (10)$$

subject to

$$N_i'(t_j) - N_i'(t_{j-1}) \leq C \qquad \text{for } i = 1, 2, \ldots m - 1;\ j - 1, 2, \ldots, n \qquad (11a)$$

$$F(t_0) = 0 \qquad (11b)$$

$$0 < t_1 < t_2 < \ldots t_n < T \qquad (11c)$$

The constraints Equation 11 parallel Equation 3 with the capacity constraint (Equation 11a) defined by Equation 9.

Up to this point the discussion assumed that the origin-destination (O-D) pattern in terms of P_{ij} is constant over time. The DP formulation above can be easily extended to handle the case where $P_{ij} = P_{ij}(t)$ even though it is difficult to estimate $P_{ij}(t)$. A reasonable approximation may be different P_{ij}'s for different periods, assuming a constant pattern within each period.

The minimum number of bus runs needed to serve the route (n_{min}), can be found by looking at the

number of buses needed to carry passengers at the maximum load point; i.e.,

$$n_{min} = \text{INT}\left\{\max_i [N_i'(t)/C]\right\} + 1 \quad (12)$$

where INT[•] is the integer value of the argument and T is the end of the design period.

The multiple O-D scheduling problem can also be solved by using a DP procedure. In a fashion similar to the last section the recursive relations that define the DP procedure are given by Equation 4 and the only difference in the execution is that the constraints (Equation 11a) have to be checked at every stage to ensure feasibility.

CASE WITH STOP DWELL TIMES DEPENDENT ON BOARDING VOLUMES

This section discusses the optimal scheduling program under the assumption that all patrons are destined to one point only (stop m in our scheme) but the dependence of the dwell time on the number of boarding passengers is taken into account. Thus it is assumed that

$$\text{Dwell time} = \Theta \cdot (\text{No. of boardings}) \quad (13)$$

where θ is a constant.

The problem in introducing this relation is that the travel time between departures from consecutive stops is not constant any more. Thus, one cannot construct the shifted demand function F(t), and each $F_i(t)$ has to be dealt with individually.

Assume that buses are dispatched at times t_j, j = 1,2,...n from 0. The jth bus arrives at stop 1 at $t_j + \Delta t_1$ to find $b_{1,j}$ passengers waiting, where

$$b_{1,j} = F_1(t_j + \Delta t_1) - F_1(t_{j-1} + \Delta t_1) \quad (14)$$

and it is delayed for $\theta \cdot b_{1,j}$ min. At stop 2 this bus picks up $b_{2,j}$ passengers, where

$$b_{2,j} = F_2(t_j + \Delta t_1 + \Theta b_{1,j} + \Delta t_2) - F_2(t_{j-2} + \Delta t_1 + \Theta b_{1,j-1} + \Delta t_2) \quad (15)$$

Let $t_{i,j}$ denote the time that the jth bus departs from the ith station. In the context of our problem this time is given by

$$t_{i,j} = t_j + \sum_{k=1}^{i} \Delta t_k + \Theta \sum_{k=1}^{i-1} b_{k,j} \quad (16a)$$

This time can be computed recursively for the jth bus, for stops i=1,2,...m by using

$$t_{i+1,j} = t_{i,j} + \Delta t_{i+1} + \Theta b_{i,j} \quad (16b)$$

The number of passengers who board the jth bus at the ith station ($b_{i,j}$) is given by the difference between the cumulative boardings between the jth and (j-1)th bus at the ith station. In other words,

$$b_{i,j} = F_i(t_{i,j}) - F_i(t_{i,j-1}) \quad (17)$$

and the capacity constraints can be now formulated as

$$\sum_i b_{i,j} \leq C \quad \text{for } j = 1, 2, \ldots, n \quad (18)$$

Given the capacity constraint, the objective functions can be formulated by using the necessary optimality condition mentioned earlier. Since each bus should pick up all the waiting passengers, the total waiting time for the patrons picked up by bus j at stop i ($w_{i,j}$) is given by

$$w_{i,j} = \int_{t_{i,j}}^{t_{i,j+1}} [F_i(t) - F_i(t_{i,j})]\, dt \quad (19)$$

In order to facilitate the presentation of the objective function, let us assume (without loss of generality) that the design period is defined such that there are no arrivals at its beginning. Accordingly, let $t_{i,o}$ define the time where $F_i(t_{i,o}) = 0$. By using these notations, the total waiting time for all buses at the ith stop ($w_i = \sum_{j=0}^{n-1} w_{i,j}$) and the total waiting time ($w = \sum_{i=1}^{m-1} w_i$). In other words,

$$w = \sum_{i=1}^{m-1} \sum_{j=0}^{n-1} \int_{t_{i,j}}^{t_{i,j+1}} [F_i(t) - F_i(t_{i,j})]\, dt \quad (20)$$

Again, the problem of minimizing Expression 20 subject to the capacity constraint (Equation 18) can be solved by using dynamic programming. The stages in the DP procedure would correspond to the dispatched buses and the states to the dispatch times. The basic (forward) DP recursion is given by

$$F_i(t_j) = \min_{t_j} [r_{j-1}(t_{j-1})] + \sum_{i=1}^{m-1} \int_{t_{i,j-1}}^{t_{i,j}} [F(t) - F(t_{i,j-1})]\, dt \quad (21)$$

A detailed numerical example of this procedure is given by Sugiyama (11), which outlines all the computational details.

Note that the approximate procedure can be very beneficial for this problem. The approximation consists of scheduling the buses based on the requirement that each bus should carry the same load. If this guideline is adopted for the problem considered here, one can follow the procedure outlined below.

Assume that each bus carries s^* passengers where $s = \sum_{i=1}^{m-1} F_i(T)/n$. The total boarding time delay is thus equal to Θs^* (assuming that each bus actually stops at each stop). If we let $\Delta\ell_i$, i= 1,2,...m denote that distance between the (i-1)th and the ith stop, the average speed of all the buses (b) is given by

$$v = \sum_{i=1}^{m} \Delta l_i \Big/ \left(\Theta s^* + \sum_{i=1}^{m} \Delta t_i\right) \quad (22)$$

The constant speed enables one now to shift $F_i(t)$ to the origin to obtain an approximate shifted cumulative demand function F(t); i.e.,

$$F(t) = \sum_{i=1}^{m} F_i\left(t + \sum_{k=1}^{i} \Delta l_k / v\right) \quad (23)$$

The solution can now be obtained by using the recursion $t_{j+1} = F^{-1}[F(t_j) + s^*]$ for j = 1,2,...n-1 where t_o is the time for which F(t) = 0.

Note that by measuring interstop bus travel times as the times between departures (as done in many studies) the ideas in this method may be in use implicitly.

SCHEDULING WITH MULTIPLE O-D AND BOARDING-DEPENDENT DWELL TIMES

This section analyzes the most general case discussed thus far--that of the optimal scheduling for a bus route with a multiple origin-destination demand pattern, assuming that the dwell times are a function of the number of boarding passengers. Again, we consider a bus route where buses are dispatched from 0 to a succession of stops i = 1,2,...m. As before, let Δt_i denote the bus travel time between the (i-1)th and the ith stop, let $F_i(t)$ denote the cumulative arrivals at stop i, and let $t_{i,j}$ denote the time that the jth bus leaves the ith stop.

The dwell time may now be a function of the number of boarding as well as alighting passengers. Let $a_{i,j}$ denote the number of passengers that alight the jth bus at the ith stop and let $d_{i,j}$ denote the dwell time of the jth bus at the ith stop. A common formula for estimating this dwell time is

$$d_{i,j} = \max(\Theta_1 a_{i,j}, \Theta_2 b_{i,j}) + E \quad (24)$$

In this expression $\theta_1 < \theta_2$ if people pay when boarding and $\theta_1 > \theta_2$ if they pay when alighting. E is a constant that represents the time it takes for the bus to pull into and out of a stop (in this paper we assume that each bus stops at all stops and thus E is included in the definition of Δt_i. This assumption holds for the problem discussed in this paper as we focus on average boarding volumes. If the average is zero for some bus stops, they should, of course, be abolished.) The number of alighting patrons can be computed from the cumulative alighting $[G_i(t)]$ as

$$a_{i,j} = G_i(t_{i,j}) - G_i(t_{i,j-1}) \quad (25a)$$

in parallel with Equation 17, which gives the number of boarding passengers. The cumulative alightings $[G_i(t)]$ can be input directly to the analysis or computed from the origin-destination matrix $(P_{i,j})$; i.e.,

$$a_{i,j} = \sum_{k=1}^{i-1} [F_k(t_{k,j}) - F_i(t_{k,j-1})] P_{k,i} \quad (25b)$$

In order to formulate the capacity constraints for the problem under study, note that

$$t_{i,j} = t_{i-1,j} + \Delta t_i + d_{i,j} \quad (26)$$

The number of passengers on the jth bus as it departs the ith station $(N_{i,j})$ is given by

$$N_{i,j} = \sum_{k=1}^{i} (b_{k,j} - a_{k,j}) \quad (27)$$

and the capacity constraint is given by

$$\underset{i=1}{\overset{m}{\mathrm{MAX}}} \left\{N_{i,j}\right\} \leq C \quad \text{for } j = 1, 2, \ldots n \quad (28)$$

The objective function is again formulated in terms of minimization of the total waiting time (w) where

$$w = \sum_{i=1}^{m-1} \sum_{j=0}^{n-1} \int_{t_{i,j}}^{t_{i,j+1}} [f_i(t) - F_i(t_{i,j})]\, dt \quad (29)$$

where $t_{i,j}$ is defined by the recursion (Equation 26). Note that the decision variables in this problem (the states of the dynamic programming) are still the dispatch times (t_j). These times are expressed implicitly in both the constraints and the objective function since

$$t_{i,j} = t_j + \sum_{k=1}^{i} \Delta t_i + \sum_{k=1}^{i-1} d_{k,j} \quad (30)$$

In the last expression $d_{k,j}$ can be expressed in terms of the inputs $[F_i(t)$ and $P_{ij}]$ as (see Equations 24, 25b, and 17)

$$d_{k,j} = \max\left\{\Theta_1 \sum_{l=1}^{k-1} [F_l(t_{l,j}) - F_l(t_{l,j-1})] P_{lk} ; \Theta_2 [F_k(t_{k,j}) - F_k(t_{k,j-1})]\right\} \quad (31)$$

Thus $t_{i,j}$ can be computed recursively given the schedule of the (j-1)th bus and the time when the jth bus left the (i-1)th stop.

The dynamic programming formulation for this problem is similar to the formulation discussed in the previous section. A numerical example under the assumption discussed is given by Sugiyama (11), who shows all the details of the DP procedure.

This deterministic optimization program can be extended to include various constraints on the minimum headway and on the time that certain buses have to visit certain stops. The problem can also be formulated in terms of buses and bus cycles rather than bus runs. This means that considerations of layover times and equipment availability can be factored into the programming of the solution.

It will be more difficult computationally to take into account the possibility of express runs, zone buses, and shortlining strategies in the optimization, even though the formulation may not pose a particular problem. The difficulty is that the number of diminsions of the state space will grow considerably under these conditions, creating a significant computational burden on the DP procedure.

OPTIMAL SCHEDULING WITH STOCHASTIC DEMAND

This section extends some aspects of the deterministic formulation of the optimal scheduling problem to include stochastic (yet time dependent) elements. The assumption is that the input demand functions are random variables that are distributed according to a Poisson probability law with time-dependent parameter $[\lambda(t)]$.

The total waiting time under these conditions (W) is a random variable and it is natural to choose the expectation of the waiting time, E[W], as the objective function to be minimized. This expectation can be decomposed as follows:

$$E[W] = E[W|A]\,\Pr(A) + E[W|\bar{A}] \cdot \Pr[\bar{A}] \quad (32)$$

where $\bar{A}$ is the event that every passenger boards the first bus that he or she sees and $\bar{A}$ is the complementary event, that some passengers are left behind at some point. The latter event is difficult to handle as it includes, for example, the case where some passenger cannot board any bus and the wait time is infinity. In general, however, the following relations hold:

$$E[W|A] < E[W|\bar{A}] \quad (33a)$$

and

$$\Pr[A] + \Pr[\bar{A}] = 1 \quad (33b)$$

Therefore, in order to minimize E[W] one can maximize Pr[A]; i.e., maximize the probability that every passenger boards the first bus. This objective function will not be capacity constrained as the capacity is included in the objective function.

Let us now analyze the simple scheduling case discussed earlier with the added assumption that the passengers' arrival rate to the ith stop follows a Poisson process with parameter $\lambda_i(t)$. This rate can be shifted to the origin stop by transforming the time by using $\lambda_i(t = \sum_{k=1}^{i} \Delta t_k)$ where Δt_k is the (deterministic) bus travel time between stop (k-1) and k. Furthermore, since arrivals at each stop are independent, the total (shifted) passenger arrivals can be described by a Poisson process with parameter $\lambda(t)$ where

$$\lambda(t) = \sum_{i=1}^{m-1} \lambda_i \left(t + \sum_{k=1}^{i} \Delta t_k\right) \quad (34)$$

If buses are dispatched at times t_j, j = 1,2,...n, the objective function is given by

$$\Pr(A) = \left[\bigcap_{j=1}^{n} A(t_{j-1}, t_j) < C \right] \quad (35a)$$

where $A(t_{j-1}, t_j)$ is the number of passengers who arrive between t_{j-1} and t_j, and t_o is the beginning of the design period. These variables are mutually independent and, therefore,

$$\Pr(A) = \prod_{j=1}^{n} \Pr[N(t_{j-1}, t_j) < C] \quad (35b)$$

The distribution of the random variable $N(t_{j-1}, t_j)$ is Poisson and thus the objective function can be expressed as

$$\Pr(A) = \prod_{j=1}^{n} \sum_{k=0}^{C} (1/k!) \left[\int_{t_{j-1}}^{t_j} \lambda(t)dt \right]^k \cdot \exp\left[\int_{t_j}^{t_{j-1}} \lambda(t)dt \right] \quad (35c)$$

Alternatively, one can maximize the logarithm of this objective function,

$$\log \Pr(A) = \sum_{j=1}^{n} \log \sum_{k=0}^{C} (1/k!) \left[\int_{t_{j-1}}^{t_j} \lambda(t)dt \right]^k \cdot \exp\left[\int_{t_j}^{t_{j-1}} \lambda(t)dt \right] \quad (36)$$

This objective function can be maximized by using a DP procedure with the recursive relations:

$$r_j(t_j) = \underset{t_j}{\mathrm{MAX}} \left\{ r_{j+1}(t_{j+1}) + \log \sum_{k=0}^{C} (1/k!) \left[\int_{t_{j-1}}^{t_j} \lambda(t)dt \right]^k \right.$$
$$\left. \mathrm{x} \exp\left[\int_{t_j}^{t_{j-1}} \lambda(t)dt \right] \right\} \quad j = n-1, n-2, \ldots 1 \quad (37)$$

where $t_k = T$. The possible dispatch instances (t_j) can be constrained by minimum and maximum headway policies to narrow the state space. Computational speed can be enhanced by using an efficient approximation for the integral.

An alternative solution method can be constructed by differentiating log Pr (A) in Equation 36 with respect to t_j, which results in the set equations:

$$\sum_{k=0}^{C} (1/k!) \left\{ \left[\int_{t_j}^{t_{j-1}} \lambda(t)dt \right]^{k-C} - \left[\int_{t_{j-1}}^{t_j} \lambda(t)dt \right]^{k-C} \right\} = 0$$
$$j = 1, 2, \ldots n-1 \quad (38)$$

This set of n-1 simultaneous equations in n-1 unknowns can be solved by Newton's method or any other algorithm for solving a set of simultaneous equations.

In parallel with the extension of the simple scheduling problem, one can extend the simple stochastic case discussed here to include a multiple O-D flow pattern and dwell times that depend on the number of boarding and alighting patrons. The extension to multiple O-D pairs is straightforward, by using the ideas expressed earlier in this paper. The other extension is somewhat more complicated analytically because some of the independence properties used in the simple problems are lost. Sugiyama (11) formulates some of these problems and suggests some solution algorithms.

Use of a more sophisticated stochastic formulation may not be very interesting because a solution to the problem when all inputs are both stochastic and time dependent does not seem feasible. The deterministic approach, however, may provide some good schedules that may actually be applied and modified, when necessary, by real time controls that are designed to contend with the randomness of transit operations.

CONCLUSION

This paper dealt with optimal scheduling of bus runs on a route. The inputs to the analysis include the passenger arrival rate as a function of time at each step, the run times between the stops, bus capacity, the O-D trip pattern (or the alighting rate at each stop), and some (estimated) parameters related to the relations between the dwell time and the number of boarding and alighting patrons. Once the problem is formulated, the optimal schedule can be found by using a dynamic programming procedure.

In its basic formulation, the solution is subject to simple capacity constraints only, but many other constraints such as service standards and nonuniform bus capacities can be incorporated easily within this framework. Further extensions were suggested.

The last section looked at some of the issues associated with modeling the arrival process as a stochastic phenomenon. Here the formulation takes on a somewhat different form as, rather than minimizing the total waiting time, the objective is to maximize the probability that each patron boards the first bus. The stochastic case is more difficult to generalize and it is concluded that deterministic methods may be the most suitable for the optimal scheduling problem.

The suggested procedures can be applied repeatedly to a given route so that a planner may trace a trade-off curve between number of buses and total passenger waiting time. Over a reasonable range of frequencies the demand for bus transport can be assumed to be inelastic to the waiting time or crowding. The abovementioned trade-off curve can thus be used for policy analysis and the setting of service standards.

In closure, we should add that the collection of detailed demand data as required by this method is an expensive and time-consuming task. Recent advances in automatic vehicle monitoring systems, however, overcome this problem. Many of these systems provide continuous information on loads and travel times that can be input to an optimization procedure.

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