Graphical Comparison of Predictions for Speed Given by Catastrophe Theory and Some Classic Models

JORGE A. ACHA-DAZA AND FRED L. HALL

Using a common data set, the speed estimates given by catastrophe theory are compared with those of some classic models. After a new proposed set of transformations to real traffic flow data, which include translation and rotation of axes, catastrophe theory is applied to information from the Queen Elizabeth Way in Ontario, Canada. The catastrophe theory model is used for predicting speeds on the basis of information on volume and occupancy. Several proposed models and a double-linear-regime model are also used for predicting speeds. The different estimates for speed are graphically compared with the observed values. A coefficient of determination (R^2) is given as the measure of reliability for all the models. The results show that the catastrophe theory model performs better than the other models.

Recent works on catastrophe theory applied to traffic flow by Navin (*1*) and Hall and others (*2–4*) have shown that traffic flow data exhibit the necessary properties for applying the cusp catastrophe surface. They have also shown that before making that application a transformation in the data is needed. Such a transformation should include a translation and rotation of axes. Developing those ideas, a way of doing those transformations is suggested by this work. Also presented in this paper are the results of the application of catastrophe theory to the transformed data. Using the same data set, other classic models are applied. Expressions for speed against flow given by Greenshields (*5*), Greenberg (*6*), and Edie (*7*), and a double-linear-regime model are also used for predicting speeds. The predictions for speed given by all the models are compared with the observed values. This comparison is made graphically and using a coefficient of determination (R^2).

The paper is organized into five sections. In the first, the data set used and the manipulation for applying the different models are described. The second section presents the elements of catastrophe theory used in this work. The way of doing the proposed translation and rotation of axes are described. In the third, catastrophe theory is applied to the data. In the fourth, the other models are applied to the data. Finally, the predictions for speed given by catastrophe theory are compared one to one against those given by the other models.

DATA

The original data set used in this work is from Station 12 median lane (left-most lane or Lane 1) of the Freeway Traffic Management System (FTMS) for the eastbound Queen Elizabeth Way (QEW) near Toronto, Canada. It consists of 2,936 thirty-sec intervals from 5 consecutive working days in the week from August 20 to 25, 1990. The location of Station 12, 2 km upstream of a recurrent morning bottleneck, and the time of the collection, from 5:30 to 10:30 a.m., made it possible to include information on both uncongested and congested regimes of traffic flow. The QEW FTMS uses paired induction loops and therefore can provide measurements of volume, occupancy, and speed.

Department of Civil Engineering, McMaster University, Hamilton, Ontario L8S 4L7, Canada.

Because regression analysis assumes an equal representation of the different zones of the data points, a random sample was drawn. This sample was taken following the procedure suggested by Drake et al. (*8*). The data were divided into intervals of 3 percent of the occupancy. Sixty points were taken from each of the 0 to 2 percent, 3 to 5 percent, 6 to 8 percent, 9 to 11 percent, 12 to 14 percent, 15 to 17 percent, and 18 to 20 percent ranges of occupancy. All 184 of the points with occupancies greater than or equal to 21 percent were also considered. These 184 points include 106 with occupancies from 21 to 30 percent, 42 from 31 to 39 percent, and 36 with occupancies greater than or equal to 40 percent. In this form, the total number of points included in the sample is 604.

CATASTROPHE THEORY

Catastrophe theory is a mathematical model that uses some special functions to represent some practical problems. The theory, created by Thom in 1972 (*9*), reproduces problems where one of the variables exhibits sudden changes and the others present smooth changes, and smooth changes might be expected in the first variable. Navin (*1*), Hall and others (*2–4*) have shown that catastrophe theory, and specifically the cusp catastrophe surface, can be applied to traffic flow, but before making that application the data have to be transformed using an axes translation and rotation.

The cusp catastrophe surface relates three variables, two known as control variables and one as the state variable. Its general expression, as given by Zeeman (*10*), is one that minimizes the potential function

$$W(X) = aX^4 + bUX^2 + cVX \qquad (1)$$

with critical surface defined by

$$4aX^3 + 2bUX + cV = 0 \qquad (2)$$

where

U, V = control variables,
X = state variable, and
a, b, c = parameters of equation.

Figure 1 presents the shape of a full cusp catastrophe surface. Points in the center part of the folded area for Figure 1 represent maxima of the potential function (Equation 1). Because the potential function is to be minimized, these points are not of interest and the center part of the fold can be removed, giving rise to what is known as the perfect delay convention.

However, there are problems where only one value occurs for the state variable in the folded area. For this case, the Maxwell convention (Figure 2) is applied. The upper and lower surfaces of the cusp catastrophe are separated by a vertical cut, and the changes from one to the other surface occur instantaneously. The potential function (Equation 1) is always at a minimum.

APPLICATIONS TO TRAFFIC FLOW

Navin (*1*) was the first researcher to apply the cusp catastrophe surface to traffic operations. Following the double-regime approach proposed by Edie (*7*), he graphically established that the orthogonal projections of a cusp catastrophe can fit the speed-volume, speed-density, and volume-density curves given by Edie's expressions. Navin selected speed as the control variable because of the sudden changes that it presents when changing from noncongested to congested regimes or from congested to noncongested. Density and volume were selected as the control variables, and the origin was related to zero operations. Navin however did not apply any real data to his formulations.

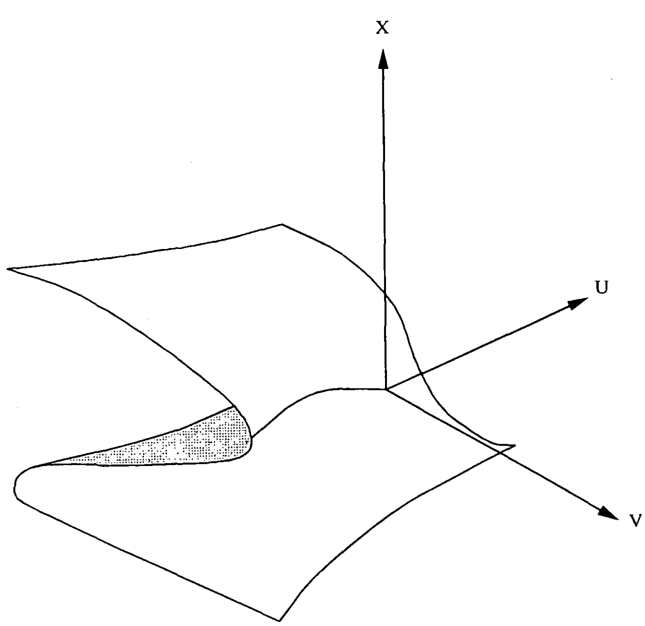

FIGURE 1 Cusp catastrophe.

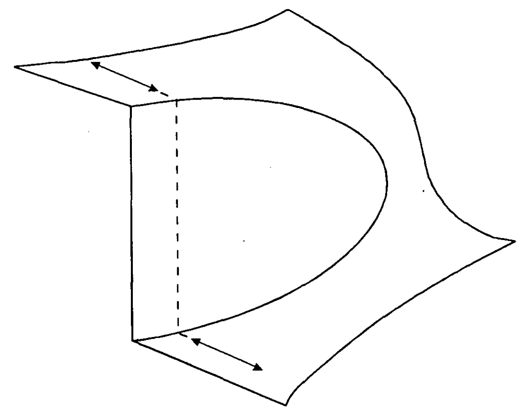

FIGURE 2 Maxwell's convention.

Hall (*2*), using real data from the QEW FTMS, drew representative curves for speed-volume, speed-occupancy, and volume-occupancy and showed graphically, like Navin, that they fit the orthogonal projections of a cusp catastrophe surface. Hall follows the same variable assignment as that of Navin. However, Hall proposed a different orientation for the axes, with the origin being associated with capacity operations. The use of representative curves meant Hall's formulation still needed to be fitted to real data. This was done in a paper by Dillon and Hall (*3*) that used a variables transformation that was in essence an axes translation. The results were not very satisfactory.

Using a visual inspection of real data sets, Forbes and Hall (*4*) compared the properties of the data with those of a cusp catastrophe surface. They found that by using the Maxwell convention, the data exhibited the necessary properties for applying catastrophe theory, but the data still needed an axes rotation before applying this convention because the orientation of the traditional axes for the volume-occupancy plane does not coincide with the position of the Maxwell convention. However, they did not carry through the calculations. Acha-Daza (*11*) followed the ideas of the previous work and applied catastrophe theory to real traffic data. Using about 50 different data sets from five stations on the QEW, he showed that the selection of a new origin with its corresponding axes translation, as proposed by Dillon and Hall (*3*), and an axes rotation for applying the Maxwell convention, as proposed by Forbes and Hall (*4*), gives results that, for some cases, may be considered to be excellent.

APPLICATION OF CATASTROPHE THEORY

The variable assignment for this work follows that of Navin (*1*) and Hall (*2*). Volume (or flow) is related to the control variable U, occupancy to the control variable V, and speed to the state variable X. The position of the origin in Figure 1 is associated for this work, as in that by Hall (*2*), with capacity operations. The origin is selected as the point of maximum volume, maximum occupancy at maximum volume, and minimum speed for maximum volume at maximum occupancy. This leads one to the axes translation given by

$$U_1 = \text{volume} - \text{maximum volume} \quad (3)$$

$$V_1 = \text{occupancy} - \text{maximum occupancy at maximum volume} \quad (4)$$

$$X_1 = \text{speed} - \text{minimum speed at maximum occupancy for maximum volume} \quad (5)$$

It was observed in the data used for this work that in some cases speed presented more than one value for different combinations of occupancy and volume. However, after a careful inspection, it was concluded that these different values were due partly to random fluctuations and partly to the precision used in the FTMS because the information for occupancy is recorded as truncated percentages. Hence the Maxwell convention can be applied.

For this analysis, the axes rotation needed in order to apply the Maxwell convention is done using the expressions

$$U = U_1 \cos \theta - V_1 \, Gf \sin \theta \quad (6)$$

$$V = U_1 \sin \theta + V_1 \, Gf \cos \theta \quad (7)$$

where Gf is a graphical factor used for solving the problem of scaling in the graphs. The use of the variables without this transformation gave distorted figures for the volume-occupancy plots, and so this parameter was introduced. Gf is given by Gf = maximum volume/maximum occupancy. The value of θ is found iteratively as the angle that minimizes the number of misclassified points, that is, those that are in the congested zone of the traffic flow but are classified to be in the uncongested, and vice versa. The limit between the zones is given by the minimum speed at maximum occupancy for maximum volume.

Once the data have been transformed, an expression of the form

$$X^3 + a_1 UX + b_1 V = 0 \quad (8)$$

is applied. In Equation 8, a_1 and b_1 are parameters that will have to be found by calibration and that are equivalent to

$$a_1 = 2b/4a \quad (9)$$

$$b_1 = c/4a \quad (10)$$

where a, b, and c are the parameters of Equation 1.

Expression 8 is used for the determination of the value of X. From this value for X, the predicted speed is found as $S_p = X$ + minimum speed at maximum occupancy for maximum volume.

The problem of the calibration of Expression 8 was solved using a direct search of the values of a_1 and b_1 that minimize the square of the difference between the observed and predicted values for speed following the Hooke and Jeeves pattern search strategy for two variables (*12*). This direct search was conducted because of the nonlinear or intrinsically linear nature of Expression 8 that does not allow one to use a traditional linear regression analysis.

Looking for a common basis of comparison among the solutions given by catastrophe theory and those of the other models, the plots of observed speed against predicted speed and a coefficient of determination, R^2, between the observed and predicted values for speed were used. The formula for R^2 was used as given by Theil (*13*,p.83).

$$R^2 = \frac{\left[\sum_{i=1}^{n} (S_{io} - \overline{S}_o)(S_{ip} - \overline{S}_p)\right]^2}{\sum_{i=1}^{n} (S_{io} - \overline{S}_o)^2 \sum_{i=1}^{n} (S_{ip} - \overline{S}_p)^2} \quad (11)$$

where

S_{io} = observed values of speed,
S_{ip} = predicted values of speed, and
$\overline{S}_o$, $\overline{S}_p$ = arithmetic means of S_{io}s and S_{ip}s.

In the different figures showing the models, R^2 is shown as R2, since the graphics program could not produce the exponents.

Using the equations just described, catastrophe theory was applied to the reduced data set. The values of the parameters for the variable transformation and for the catastrophe expression used were as follows: maximum volume = 28 veh/30 sec, maximum occupancy at maximum volume = 15 percent, minimum speed at maximum occupancy for maximum volume = 95 km/hr, maximum occupancy = 81 percent, angle of rotation = −7.8 degrees. The application of the calibration process gave value for a_1 = −120 and b_1 = 44,413. The coefficient of determination (R^2) using these values of the parameters is equal to 0.92. The graph of observed against predicted values for speed is shown in Figure 3.

OTHER MODELS CONSIDERED

Because the rest of the models are intrinsically linear or linear in parameters, a least-squares linear regression method was used for the determination of their parameters. The expressions used for the calibration and the values of the parameters are presented in the following.

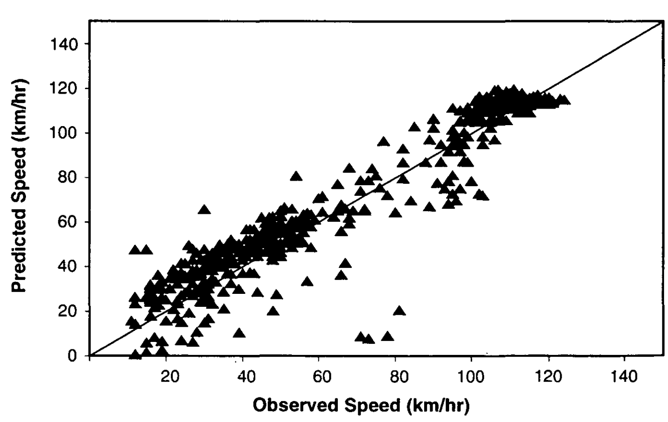

FIGURE 3 Speeds estimated using catastrophe theory plotted against observed speeds.

Greenshields' Model

For the application of the Greenshields' model the expression used was

$$Q = K_j V - (K_j/V_f)V^2 \quad (12)$$

where

Q = flow (veh/hr);
K_j = a constant, jam density (veh/km);
V = speed (km/hr); and
V_f = a constant, free-flow speed (km/hr).

The calibration for the data set considered gave the values of parameters

$$Q = 60.9V - 0.432V^2 \quad (13)$$

The predicted values for speed were calculated using the observed values for the flow (vehicles per hour) and solving the quadratic Equation 13 for V. The calculation of the coefficient of determination (R^2) gave a value of 0.87. The plot of the observed values of speed against those predicted by Greenshields' model is presented in Figure 4. It is important to note that the value of the R^2 corresponds to a reduced sample because values of flow over 2,160 veh/hr make the values of V in Expression 13 imaginary.

Greenberg's Model

Greenberg's model was applied using the expression

$$\ln(Q/V) = \ln(K_j) - V/c \quad (14)$$

where c is the speed at capacity (km/hr). After the determination of the parameters, Expression 14 becomes

$$\ln(Q/V) = 4.41 - 0.0177V \quad (15)$$

The predicted values for speed were calculated using the observed values for the flow and solving for V in Equation 15. The value for the R^2 in this model was 0.611. The plot of the observed values of speed against the calculated speeds is presented in Figure 5.

Edie's Model

Before applying Edie's model and also the two-linear-regime model, the data set was divided in two subsets, one for the uncongested data and the other for the congested points. The critical value used for this division was the minimum speed at maximum occupancy for maximum volume (95 km/hr). The same expression used for Greenberg's model was used for the congested data. For the uncongested data the expression used was

$$Q = K_c \ln(V_f)V - K_c V \ln(V) \quad (16)$$

where K_c is the density at capacity in vehicles per kilometer, a constant. The application of the least-squares regression led to the expressions

$$\ln(Q/V) = 4.11 - 0.0108V \quad (17)$$

for the congested zone, and

$$Q = 515V - 107V\ln(V) \quad (18)$$

for the uncongested zone.

The observed values of flow were used in Equations 17 and 18 to determine the predicted speeds. These predicted speeds were plotted against their observed values (Figure 6). The combined R^2 for both zones is equal to 0.80.

Double-Linear-Regime Model

Greenshields' expression was used twice for this model, one for the uncongested zone and the other for the congested one. For the uncongested zone the expression found is

$$Q = 121V - 0.983V^2 \quad (19)$$

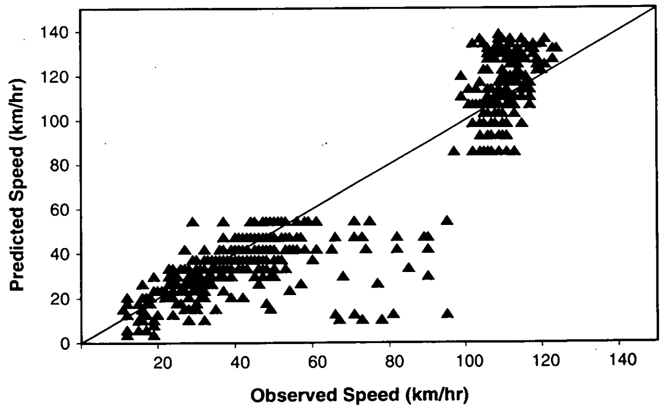

FIGURE 4 Speeds estimated using Greenshield's model plotted against observed speeds.

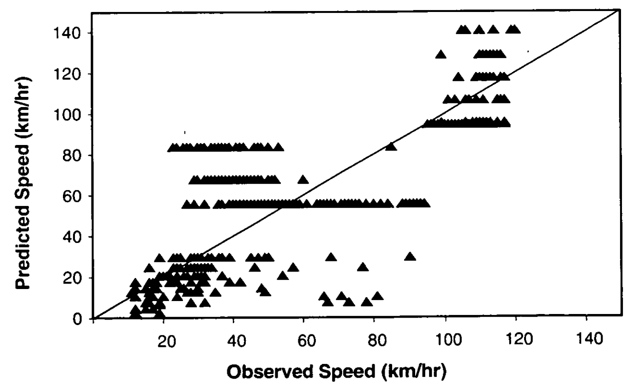

FIGURE 5 Speeds predicted using Greenberg's model plotted against observed speeds.

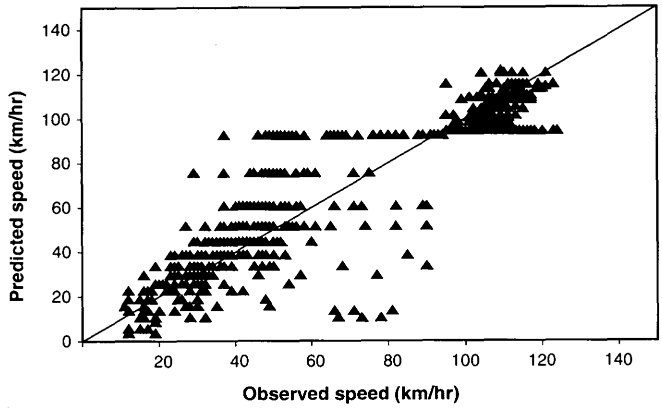

FIGURE 6 Speeds predicted using Edie's model plotted against observed speeds.

and for the congested zone the expression is

$$Q = 54.8V - 0.347V^2 \quad (20)$$

The observed flows were used in Equations 19 and 20 to calculate the predicted speeds. The observed values of speed were plotted against those calculated speeds (Figure 7). The value of the combined R^2 is 0.89. Note that the solutions for V of Equation 19 are imaginary roots for values of the volume over 2,040 veh/hr and for Equation 20 for values of the volume over 1,920 veh/hr. This reduced the sample size for this model to 570 points.

COMPARISON OF CATASTROPHE THEORY AGAINST OTHER MODELS

When observing Figures 3 to 7, several characteristics are found. First, only catastrophe theory can react to changes in concentration. For constant values of the volume, the other models predict constant speeds without considering differences in density. Second, the predictions for speed given by Greenberg's model are well beyond the limit of 150 km/hr. The solutions given by this model are the worst. Third, the double-linear-regime model does not predict intermediate values for speed. It clearly separates the data in two different zones with speeds well above or below the critical value considered (95 km/hr).

Regarding the values of the coefficient of determination (R^2), catastrophe theory gives the best value even when considering the problem of the reduced sample size for Greenshields' and the double-linear models. If only the reduced samples of those two models are considered, the R^2 values for catastrophe theory remain at 0.92 in both cases.

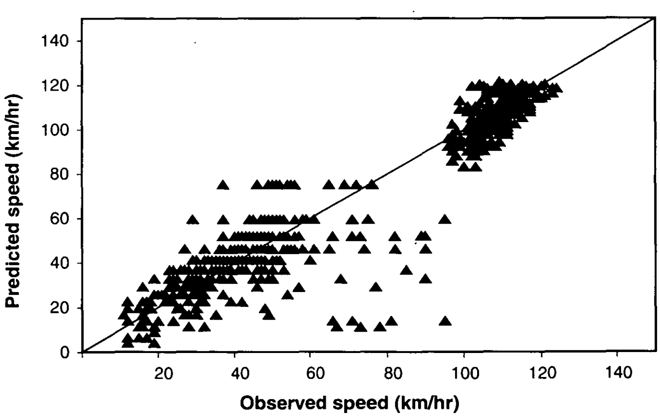

FIGURE 7 Speeds predicted using double-linear-regime model plotted against observed speeds.

CONCLUSION

Solely on the basis of the graphical comparison of Figures 3 to 7 and on the calculated values of R^2, catastrophe theory can be claimed to be the best model. The proposed transformation in the variables and axes works adequately for this case. The application of catastrophe theory to traffic flow has become a reality. It can reproduce the relationship among traffic flow variables better than the other models considered. An additional advantage of catastrophe theory is that it eliminates the necessity for the use of a double-regime model such as Edie's or the double-linear one.

The results presented can lead one to think of catastrophe theory as a model that will give excellent solutions when applied to traffic flow operations. However, this is not always the case. When applying catastrophe theory to some different data sets used in his thesis, Acha-Daza (*11*) found that the solutions given by this model were not necessarily as good as in the example presented in this paper. Data from stations affected by queue discharge flow (i.e., locations where drivers are accelerating, with consequent nonequilibrium values for speed) suggested that the application of catastrophe theory to these cases is not appropriate. The predicted values for speed were nearly constant around two values, one for the congested zone (queue discharge data) and one for the uncongested zone. At those stations, the model did not react adequately to changes in volume or occupancy.

The excellent results given by catastrophe theory can be explained only by the nature of the traffic operations in multilane highways. The correct representation of the operations on these highways is very much closer to a behavioral problem than to a physical one. Those are the kind of situations that catastrophe theory can model and for which its earliest applications were done.

The catastrophe theory model was found to be very susceptible to changes in the values of the parameters used for the axes translation (maximum volume, occupancy at maximum volume, and minimum speed at maximum occupancy for maximum volume). Changes in those parameters can make the solutions for the same data set very different. Probably it is necessary to define those parameters in a different way, trying to identify some constant values rather than defining them only from the observed data. Further work can determine if that will give better solutions.

ACKNOWLEDGMENTS

Most of the research for this paper was performed during studies for Jorge A. Acha-Daza's master's degree in engineering, financial support for which through the McMaster-CIDA Scholarship program is gratefully acknowledged. The authors also wish to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada and the assistance of the Ministry of Transportation of Ontario in making the data available.

REFERENCES

1. F. P. D. Navin. Traffic Congestion Catastrophes. *Transportation Planning and Technology*, Vol. 11, 1988, pp. 19–25.
2. F. L. Hall. An Interpretation of Speed-Flow Concentration Relationships Using Catastrophe Theory. *Transportation Research-A*, Vol. 21A, No. 3, 1987, pp. 191–201.
3. D. S. Dillon and F. L. Hall. Freeway Operations and the Cusp Catastrophe: An Empirical Analysis. In *Transportation Research Record 1132*, TRB, National Research Council, Washington, D.C., 1988, pp. 66–76.
4. G. J. Forbes and F. L. Hall. The Applicability of Catastrophe Theory in Modelling Freeway Traffic Operations. *Transportation Research-A*, Vol. 24-A, No. 5, 1990, pp. 335–344.
5. B. D. Greenshields. A Study in Highway Capacity. *Proc.*, Vol. 14, 1934, pp. 448–477.
6. H. Greenberg. An Analysis of Traffic Flow. *Operations Research*, Vol. 7, 1959, pp. 78–85.
7. L. C. Edie. Car Following and Steady-State Theory for Non-Congested Traffic. *Operations Research*, Vol. 9, 1961, pp. 66–76.
8. J. S. Drake, J. L. Schofer, and A. D. May. A Statistical Analysis of Speed Density Hypothesis. In *Highway Research Record 154*, HRB, National Research Council, Washington, D.C., 1967, pp. 53–87.
9. R. Thom. *Structural Stability and Morphogenesis* (D. H. Fowler, translator). Benjamin, Reading, Mass., 1975.
10. E. C. Zeeman. *Catastrophe Theory, Selected Papers, 1972–1977.* Addison-Wesley, London, England, 1977.
11. J. A. Acha-Daza. *The Application of Catastrophe Theory to Traffic Flow Variables*. M. Eng. thesis. Department of Civil Engineering, McMaster University, Hamilton, Ontario, Canada, 1992.
12. A. A. Smith, E. Hinton, and R. W. Lewis. *Civil Engineering Systems Analysis and Design*. John Wiley and Sons, New York, N.Y., 1983, pp. 183–185.
13. H. Theil. *Introduction to Econometrics*. Prentice-Hall, Englewood Cliffs, N.J., 1978.

Publication of this paper sponsored by Committee on Traffic Flow Theory and Characteristics.