Tabulated Values for Determining the Complete Pattern of Stresses, Strains, and Deflections Beneath a Uniform Circular Load on a Homogeneous Half Space

R. G. AHLVIN and H. H. ULERY, respectively, Chief and Engineer, Special Projects Section, Flexible Pavement Branch, Soils Division, U. S. Army Engineer Waterways Experiment Station, Vicksburg

Results of extensive computations are presented in a manner that permits simple determination of the complete pattern of stress, strain, and deflection at points beneath a uniform circular load on an elastic, homogeneous, isotropic, half space. Tabulations are presented of eight factors for a dense network of points in terms of depth below the load and radial distance from the load axis. Simple formulas are presented using these factors and Poisson's ratio by which any coordinate stress, strain, or deflection may be determined for any value of Poisson's ratio.

•THIS PAPER presents results of computations of stress, strain, and deflection at points beneath a uniform circular load on an elastic, homogeneous, isotropic half space. These results are being reported in the hope that they will provide a useful reference to anyone concerned with the distribution of stress and strain induced by tire or footing loads that may be treated as uniformly distributed circular loads. Computations of this type are not new (1, 2, 3); but the results presented are somewhat more extensive, provide a denser network of values, give greater accuracy, and treat Poisson's ratio more directly than previously published information. The method of presentation is also somewhat different in that tabulations have been made of each of eight functions which can be combined through simple equations to give any coordinate stress, strain, or deflection for any value of Poisson's ratio.

The computations reported were made in connection with an investigation of pressures and deflections in homogeneous soil masses. This was part of a larger study of the action of loads on flexible pavements conducted at the U. S. Army Engineer Waterways Experiment Station. Theoretical values of stress, strain, and deflection were developed for comparison with equivalent values measured in field test sections (7, 8, 9). These theoretical values were computed using equations developed by Love (5) and other methods (8 and 9).

In the computations, certain functions were obtained that repeat themselves and are independent of Poisson's ratio. These are the functions that are tabulated and presented which combine in simple formulas to give stress, strain, or deflection for any value of Poisson's ratio. Tables 1 through 8 give the data in terms of depth below the loaded area and offset distance from the center of the loaded area, of the eight functions needed to determine any desired stress, strain, or deflection. The tables give values for depths in radii of 0, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5, 6, 7, 8, and 9 for offset distances from the load axis in radii of 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, and 14. In addition, values are presented for depths in radii of 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.2, 1.5, 2, 2.5, 3, and 4 at offset distances in radii of 0.2, 0.4, 0.6, 0.8, 1.2, and 1.5. A few values are included for depths of 10 radii and a few for fractional radii depths at other offset distances. The values included represent those needed for comparison with equivalent values measured in test sections as mentioned previously.

It might appear strange that, with the modern computing equipment now available, tabular results of the type reported are not already commonly available.

Table 1

Function "A"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0.2	0 4	0.6	0 8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	1 0	1.0	1 0	1 0	1.0	5	0	0	0	0	0	0	0	0	0	0	0	0
0.1	90050	.89748	88679	86126	.78797	.43015	09645	02787	00856	.00211	00084	00042						
0.2	80388	79824	77884	73483	63014	.38269	15433	05251	01680	.00419	.00167	00083	00048	00030	00020			
0.3	.71265	.70518	.68316	.62690	52081	34375	.17964	.07199	02440	.00622	.00250							
0.4	62861	62015	59241	.53767	.44329	31048	18709	.08593	03118									
0.5	.55279	54403	51622	46448	38390	28156	18556	.09499	.03701	01013	00407	.00209	00118	00071	00053	00025	.00014	00009
0.6	48550	47691	45078	40427	33676	25588	17952	.10010										
0.7	42654	.41874	.39491	35428	29833	21727	17124	.10228	04558									
0 8	37531	36832	34729	31243	.26581	21297	16206	.10236										
0 9	33104	32492	30669	27707	23832	19488	15253	.10094										
1	29289	28763	27005	24697	.21468	17868	14329	09849	05185	01742	00761	00393	00226	00143	00097	00050	.00029	00018
1.2	23178	22795	21662	19890	.17626	15101	12570	.09192	05260	.01935	.00871	00459	00269	00171	00115			
1.5	16795	16552	15877	14804	.13436	.11892	10296	08048	.05116	02142	01013	00548	.00325	.00210	00141	00073	00043	00027
2	10557	.10453	10140	09647	09011	08269	07471	06275	04496	.02221	.01160	00659	00399	00264	00180	.00094	00056	00036
2.5	07152	07098	06947	06698	.06373	05974	.05555	.04880	03787	.02143	.01221	00732	.00463	.00308	00214	00115	.00068	00043
3	05132	05101	.05022	.04886	.04707	.04487	.04241	.03839	03150	.01980	.01220	00770	.00505	00346	00242	00132	00079	00051
4	.02986	.02976	.02907	.02802	.02832	02749	02651	.02490	02193	01592	.01109	00768	00536	00384	.00282	00160	00099	00065
5	01942	01938				.01835			01573	.01249	.00949	00708	00527	.00394	00298	00179	00113	00075
6	01361					01307			01168	.00983	.00795	00628	00492	00384	00299	00188	00124	00084
7	01005					00976			00894	00784	.00661	00548	00445	00360	00291	00193	00130	00091
8	.00772					00755			.00703	.00635	00554	00472	00398	00332	00276	00189	00134	00094
9	00612					00600			00566	00520	.00466	00409	00353	00301	00256	00184	00133	00096
10								00477	00465	00438	00397	00352	00326	00273	00241			

Table 2

Function "B"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0 2	0 4	0 6	0 8	1	1 2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	09852	10140	11138	13424	18796	05388	-.07899	- 02672	- 00845	- 00210	- 00084	- 00042						
0.2	18857	19306	20772	23524	25983	08513	- 07759	- 04448	- 01593	- 00412	-.00166	- 00083	- 00024	- 00015	- 00010			
0.3	26362	26787	.28018	29483	27257	10757	- 04316	-.04999	- 02166	- 00599	-.00245							
0.4	32016	32259	32748	32273	26925	12404	- 00766	- 04535	- 02522									
0.5	35777	35752	.35323	33106	26236	13591	.02165	-.03455	- 02651	-.00991	- 00388	- 00199	- 00116	- 00073	- 00049	- 00025	- 00014	- 00009
0.6	37831	37531	.36308	32822	25411	14440	04457	-.02101										
0.7	38487	37962	36072	31929	24638	.14986	06209	-.00702	- 02329									
0.8	.38091	37408	.35133	30699	.23779	.15292	.07530	.00614										
0.9	36962	36275	.33734	29299	22891	.15404	.08507	01795										
1	35355	.34553	.32075	27819	.21978	15355	09210	02814	- 01005	- 01115	- 00608	- 00344	- 00210	-.00135	- 00092	- 00048	- 00028	- 00018
1.2	.31485	30730	28481	24836	.20113	14915	10002	04378	00023	- 00995	-.00632	- 00378	- 00236	- 00156	- 00107			
1.5	25602	25025	23338	20694	17368	13732	10193	05745	01385	- 00669	-.00600	- 00401	- 00265	- 00181	- 00126	- 00068	- 00040	- 00026
2	17889	18144	16644	15198	13375	11331	09254	.06371	02836	00028	-.00410	- 00371	- 00278	- 00202	- 00148	- 00084	- 00050	- 00033
2.5	12807	12633	12126	11327	.10298	09130	07869	06022	03429	00661	- 00130	-.00271	- 00250	- 00201	- 00156	- 00094	- 00059	- 00039
3	09487	09394	09099	08635	08033	07325	06551	05354	.03511	01112	.00157	- 00134	- 00192	- 00179	- 00151	- 00099	- 00065	- 00046
4	05707	05666	.05562	05383	05145	04773	04532	.03995	03066	01515	00595	00155	- 00029	- 00094	- 00109	- 00094	- 00068	- 00050
5	03772	03760				03384			02474	01522	00810	00371	00132	00013	- 00043	- 00070	- 00061	- 00049
6	02666					02468			01968	01380	00867	00496	00254	00110	00028	- 00037	- 00047	- 00045
7	01980					01868			01577	01204	.00842	00547	.00332	00185	00093	- 00002	- 00029	- 00037
8	01526					01459			01279	01034	00779	00554	00372	00236	.00141	00035	- 00008	- 00025
9	01212					.01170			.01054	00888	00705	00533	.00386	00265	00178	00066	00012	- 00012
10								00924	00879	00764	.00631	00501	.00382	00281	00199			

Table 3

Function "C"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0.2	0.4	0 6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	-.04926	-.05142	-.05903	-.07708	-.12108	.02247	.12007	.04475	.01536	.00403	.00164	.00082						
0.2	-.09429	-.09755	-.10872	-.12977	- 14552	.02419	.14896	.07892	.02951	.00796	.00325	00164	.00094	00059	.00039			
0.3	-.13181	-.13484	-.14415	-.15023	-.12990	.01988	.13394	.09816	.04148	.01169	.00483							
0.4	-.16008	-.16188	-.16519	-.15985	- 11168	.01292	.11014	.10422	.05067									
0.5	-.17889	-.17835	-.17497	-.15625	-.09833	.00483	.08730	.10125	.05690	.01824	.00778	00399	.00231	.00146	.00098	.00050	.00029	.00018
0.6	-.18915	-.18664	-.17336	-.14934	-.08967	-.00304	.06731	.09313										
0.7	-.19244	-.18831	-.17393	-.14147	-.08409	-.01061	.05028	.08253	.06129									
0.8	- 19046	-.18481	-.16784	-.13393	- 08066	-.01744	.03582	.07114										
0.9	-.18481	-.17841	-.16024	-.12664	-.07828	-.02337	.02359	.05993										
1	-.17678	-.17050	-.15188	-.11995	- 07634	-.02843	.01331	.04939	.05429	.02726	.01333	.00726	00433	.00278	00188	.00098	00057	.00036
1.2	-.15742	-.15117	-.13467	-.10763	-.07289	-.03575	-.00245	.03107	.04552	.02791	.01467	.00824	.00501	.00324	.00221			
1.5	-.12801	-.12277	-.11101	-.09145	-.06711	-.04124	-.01702	.01088	03154	.02652	.01570	.00933	00585	00386	00266	.00141	00083	00039
2	- 08944	-.08491	-.07976	-.06925	-.05560	-.04144	-.02687	-.00782	01267	.02070	.01527	.01013	.00321	00462	00327	.00179	00107	00069
2.5	-.06403	-.06068	-.05839	-.05259	-.04522	-.03605	-.02800	-.01536	.00103	.01384	.01314	00987	.00707	.00506	.00369	00209	00128	00083
3	- 04744	-.04560	-.04339	-.04089	-.03642	-.03130	- 02587	-.01748	-.00528	.00792	.01030	.00888	.00689	.00520	.00392	.00232	.00145	.00096
4	-.02854	-.02737	-.02562	- 02585	-.02421	- 02112	-.01964	-.01586	-.00956	.00038	.00492	00602	.00561	.00476	00389	00254	.00168	00115
5	- 01886	- 01810				-.01568			-.00939	- 00293	-.00128	.00329	.00391	00380	00341	00250	00177	00127
6	-.01333					-.01118			-.00819	-.00405	-.00079	00129	00234	.00272	00272	00227	00173	.00130
7	-.00990					-.00902			-.00678	-.00417	-.00180	- 00004	.00113	.00174	00200	00193	00161	00128
8	-.00763					-.00699			- 00552	-.00393	-.00225	-.00077	.00029	.00096	.00134	.00157	.00143	.00120
9	-.00607					-.00423			-.00452	-.00353	-.00235	-.00118	- 00027	.00037	00082	00124	00122	00110
10								-.00381	-.00373	-.00314	-.00233	-.00137	-.00063	.00030	.00040			

Table 4

Function "D"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	.04926	.04998	.05235	.05716	.06687	.07635	.04108	.01803	.00691	.00193	.00080	.00041						
0.2	.09429	09552	.09900	10546	.11431	.10932	.07139	.03444	.01359	.00384	.00159	.00081	.00047	.00029	00020			
0.3	13181	.13305	.14051	.14062	.14267	.12745	.09078	.04817	01982	.00927	.00238							
0.4	.16008	.16070	.16229	.16288	.15756	.13696	.10248	.05887	.02545									
0.5	17889	.17917	.17826	.17481	16403	.14074	.10894	.06670	.03039	00921	.00390	.00200	00116	.00073	.00049	00025	00015	00009
0.6	.18915	.18867	18573	.17887	.16489	14137	11186	.07212										
0.7	19244	.19132	.18679	17782	.16229	13926	11237	.07551	.03801									
0.8	.19046	.18927	18348	.17306	.15714	13548	11115	.07728										
0.9	.18481	.18349	17709	16635	.15063	.13067	10866	.07788										
1	17678	.17503	.16886	.15824	14344	12513	.10540	.07753	.04456	.01611	.00725	00382	.00224	.00142	00096	.00050	00029	00018
1.2	.15742	.15618	15014	14073	12823	11340	09757	.07484	.04575	.01796	.00835	.00446	.00264	00169	00114			
1.5	12801	12754	.12237	11549	.10657	09608	08491	.06833	.04539	.01983	.00970	.00532	.00320	00205	.00140	00073	.00043	00027
2	08944	.09080	.08668	08273	.07814	.07187	06566	.05589	.04103	02098	.01117	.00643	.00398	.00260	.00179	00095	00056	.00036
2.5	.06403	.06565	06284	.06068	05777	.05525	.05069	.04486	.03532	02045	.01183	00717	00457	.00306	.00213	.00115	00068	.00044
3	.04744	.04834	.04760	.04548	.04391	.04195	.03963	.03606	.02983	.01904	.01187	.00755	.00497	00341	.00242	00133	.00080	00052
4	.02854	.02928	.02996	.02798	.02724	.02661	.02568	.02408	.02110	.01552	.01087	00757	00533	.00382	00280	.00160	00100	.00065
5	01886	.01950				.01816			.01535	.01230	.00939	.00700	.00523	.00392	00299	00180	.00114	.00077
6	.01333					.01351			.01149	.00976	.00788	.00625	.00488	00381	.00301	.00190	00124	.00086
7	.00990					.00966			00899	.00787	.00662	.00542	00445	00360	00292	.00192	00130	.00092
8	00763					.00759			.00727	.00641	.00554	00477	.00402	.00332	.00275	.00192	00131	.00096
9	.00607					00746			.00601	.00533	.00470	.00415	.00358	00303	00260	.00187	.00133	00099
10								.00542	00506	.00450	.00398	.00364	.00319	00278	00239			

Table 5

Function "E"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0 2	0 4	0 6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	5	5	.5	.5	.5	5	34722	.22222	12500	05556	03125	02000	.01389	01020	.00781	.00500	.00347	00255
0.1	45025	449494	.44698	.44173	.43008	.39198	30445	20399	.11806	.05362	.03045	.01959						
0.2	40194	400434	.39591	.38660	36798	32802	26598	.18633	.11121	.05170	.02965	.01919	01342	.00991	00762			
0.3	35633	.35428	.33809	33674	.31578	28003	.23311	.16967	.10450	.04979	.02886							
0.4	.31431	31214	.30541	.29298	27243	24200	.20526	.15428	.09801									
0.5	.27639	.27407	.26732	.25511	.23639	21119	18168	14028	09180	04608	.02727	.01800	01272	.00946	00734	00475	.00332	.00246
0.6	.24275	.24247	.23411	.22289	.20634	.18520	.16155	.12759										
0.7	.21327	.21112	.20535	.19525	.18093	16356	.14421	.11620	.08027									
0.8	.18765	.18550	.18049	17190	.15977	.14523	.12928	.10602										
0.9	.16552	.16337	.15921	.15179	.14168	.12954	.11634	.09686										
1	.14645	.14483	.14610	.13472	.12618	.11611	.10510	.08865	.06552	.03736	.02352	.01602	.01157	.00874	.00683	00450	.00318	.00237
1.2	.11589	.11435	.11201	.10741	.10140	.09431	.08657	.07476	.05728	.03425	.02208	.01527	.01113	00847	.00664			
1.5	.08398	.08356	.08159	.07885	.07517	.07088	.06611	.05871	.04703	.03003	.02008	01419	.01049	00806	.00636	.00425	.00304	.00228
2	.05279	05105	.05146	.05034	.04850	.04675	.04442	.04078	.03454	.02410	.01706	.01248	.00943	.00738	00590	.00401	.00290	00219
2.5	.03576	.03426	.03489	.03435	.03360	.03211	.03150	.02953	02599	.01945	.01447	.01096	.00850	00674	.00546	00378	.00276	00210
3	.02566	.02519	.02470	.02491	.02444	.02389	.02330	.02216	.02007	.01585	.01230	.00962	.00763	.00617	00505	.00355	00263	00201
4	.01493	01452	.01495	.01526	.01446	.01418	.01395	.01356	.01281	.01084	.00900	00742	.00612	00511	00431	00313	00237	.00185
5	.00971	00927				.00929			.00873	.00774	.00673	.00579	.00495	00425	.00364	.00275	.00213	.00168
6	.00680					.00632			.00629	00574	.00517	00457	.00404	00354	00309	00241	00192	00154
7	.00503					.00493			.00466	.00438	.00404	.00370	.00330	.00296	.00264	.00213	.00172	00140
8	.00386					.00377			.00354	.00344	.00325	.00297	.00273	.00250	.00228	.00185	00155	.00127
9	.00306					.00227			00275	00273	.00264	.00246	.00229	.00212	.00194	.00163	00139	00116
10								00210	.00220	.00225	00221	.00203	.00200	.00181	.00171			

Table 6

Function "F"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0 2	0 4	0 6	0 8	1	1 2	1.5	2	3	4	5	6	7	8	10	12	14
0	5	.5	5	5	5	0	-.34722	-.22222	-.12500	-.05556	-.03125	-.02000	- 01389	- 01020	- 00781	- 00500	-.00347	- 00255
0.1	45025	.44794	43981	41954	.35789	03817	-.20800	-.17612	-.10950	-.05151	- 02961	- 01917						
0 2	40194	39781	38294	34823	26215	05466	- 11165	-.13381	- 09441	- 04750	-.02798	-.01835	-.01295	-.00961	- 00742			
0.3	.35633	.35094	34508	29016	20503	06372	- 05346	- 09768	- 08010	-.04356	-.02636							
0.4	31431	30801	.28681	.24469	.17086	.06848	-.01818	-.06835	- 06684									
0.5	.27639	.26997	24890	20937	.14752	07037	00388	- 04529	-.05479	- 03595	-.02320	-.01590	-.01154	- 00875	- 00681	-.00450	-.00318	-.00237
0.6	24275	.23444	21667	18138	13042	07068	01797	- 02749										
0.7	21327	.20762	.18956	.15903	11740	.06963	.02704	-.01392	-.03469									
0.8	18765	.18287	16679	14053	10604	06774	03277	- 00365										
0.9	16552	16158	14747	.12528	.09664	.06533	.03619	00408										
1	.14645	.14280	12395	11225	.08850	.06256	03819	.00984	- 01367	-.01994	-.01591	-.01209	-.00931	-.00731	- 00587	-.00400	- 00289	-.00219
1.2	11589	.11360	10460	09449	07486	05670	03913	.01716	- 00452	- 01491	-.01337	- 01068	- 00844	-.00676	- 00550			
1.5	08398	08196	.07719	.06918	.05919	04804	03686	.02177	.00413	- 00879	-.00995	- 00870	-.00723	- 00596	- 00495	-.00353	- 00261	- 00201
2	05279	05348	04994	04614	04162	03593	03029	02197	.01043	-.00189	-.00546	- 00589	-.00544	- 00474	-.00410	-.00307	- 00233	-.00183
2.5	03576	.03673	.03459	03263	.03014	02762	02406	01927	01188	00198	-.00226	- 00364	-.00386	- 00366	- 00332	- 00263	- 00208	- 00166
3	02566	.02586	02255	02395	02263	02097	.01911	01623	.01144	.00396	-.00010	- 00192	-.00258	- 00271	- 00263	- 00223	- 00183	- 00150
4	01493	.01536	.01412	01259	01386	.01331	.01256	.01134	.00912	.00508	.00209	.00026	- 00076	- 00127	-.00148	- 00153	-.00137	-.00120
5	.00971	01011				00905			00700	.00475	00277	00129	.00031	-.00030	- 00066	- 00096	- 00099	- 00093
6	00680					00675			00538	00409	.00278	.00170	00088	00030	- 00010	- 00053	-.00066	-.00070
7	00503					00483			00428	00346	.00258	00178	00114	.00064	00027	-.00020	- 00041	- 00049
8	00386					00380			.00350	.00291	.00229	.00174	.00125	00082	.00048	.00003	-.00020	-.00033
9	00306					00374			00291	00247	.00203	.00163	00124	00089	00062	00020	- 00005	- 00019
10								00267	00246	.00213	.00176	00149	.00126	00092	00070			

Table 7

Function "G"

Depth (Z) in Radii	Offset (ρ) in Radii																	
	0	0.2	0.4	0.6	0.8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	0	0	0	0	0	.31831	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0	.00315	.00802	.01951	.06682	.31405	.05555	.00865	.00159	.00023	.00007	.00003						
0.2	0	.01163	.02877	.06441	.16214	.30474	.13592	.03060	.00614	.00091	.00026	.00010	.00005	00003	.00002			
0.3	0	.02301	.05475	.11072	.21465	.29228	.18216	05747	.01302	00201	.00059							
0.4	0	.03460	.07883	.14477	23442	.27779	.20195	.08233	.02138									
0.5	0	.04429	.09618	.16426	.23652	.26216	.20731	.10185	03033	.00528	.00158	00063	.00030	00016	00009	.00004	00002	00001
0.6	0	.04966	.10729	17192	.22949	.24574	.20496	.11541										
0.7	0	.05484	.11256	.17126	.21772	.22924	19840	.12373	04718									
0.8	0	.05590	.11225	.16534	.20381	.21295	.18953	.12855										
0.9	0	.05496	.10856	.15628	.18904	.19712	.17945	.28881										
1	0	.05266	.10274	.14566	.17419	.18198	.16884	.12745	06434	.01646	.00555	.00233	.00113	.00062	00036	.00015	.00007	.00004
1.2	0	.04585	.08831	.12323	.14615	.15408	14755	.12038	.06967	.02077	.00743	.00320	.00159	.00087	00051			
1.5	0	.03483	.06688	.09293	.11071	.11904	.11830	.10477	.07075	.02599	.01021	.00460	.00233	00130	00078	.00033	00016	.00009
2	0	.02102	.04069	.05721	.06948	.07738	.08067	.07804	.06275	.03062	.01409	.00692	00369	.00212	00129	00055	00027	.00015
2.5	0	.01293	.02534	.03611	.04484	.05119	.05509	.05668	.05117	.03099	.01650	.00886	.00499	.00296	.00185	.00082	.00041	.00023
3	0	.00840	.01638	.02376	.02994	.03485	.03843	.04124	.04039	.02886	.01745	.01022	.00610	.00376	.00241	.00110	.00057	00032
4	0	.00382	.00772	.01149	.01480	.01764	.02004	.02271	.02475	.02215	.01639	.01118	.00745	.00499	00340	00167	.00090	00052
5	0	.00214				.00992		.01343	.01551	.01601	.01364	.01105	.00782	.00560	.00404	.00216	.00122	00073
6	0					.00602		.00845	.01014	01148	.01082	00917	00733	.00567	.00432	.00243	.00150	00092
7	0					.00396			00687	.00830	.00842	00770	.00656	00539	00432	.00272	.00171	.00110
8	0					.00270			.00481	.00612	.00656	.00631	.00568	00492	00413	00278	00185	.00124
9	0					.00177			00347	.00459	.00513	.00515	.00485	00438	.00381	00274	00192	00133
10	0							.00199	.00258	.00351	.00407	.00420	00411	.00382	00346			

Table 8

Function "H"

Depth (Z) in Radii	Offset (ρ) in Radii																
	0	0.2	0.4	0.6	0 8	1	1.2	1.5	2	3	4	5	6	7	8	10	12	14
0	2 0	1 97987	1.91751	1.80575	1.62553	1.27319	.93676	.71185	51671	.33815	.25200	20045	.16626	14315	12576	.09918	08346	07023
0.1	1 80998	1.79018	1.72886	1.61961	1 44711	1.18107	.92670	.70888	51627	.33794	.25184	20081						
0.2	1.63961	1.62068	1.56242	1 46001	1 30614	1 09996	.90098	70074	51382	.33726	.25162	.20072	16688	14288	12512			
0.3	1.48806	1.47044	1 40979	1 32442	1 19210	1.02740	.86726	.68823	.50966	33638	.25124							
0.4	1.35407	1.33802	1.28963	1.20822	1.09555	.96202	.83042	.67238	.50412									
0.5	1.23607	1.22176	1.17894	1.10830	1.01312	.90298	79308	65429	.49728	.33293	.24996	.19982	.16668	.14273	.12493	09996	08295	.07123
0.6	1 13238	1.11998	1.08350	1.02154	94120	.84917	.75653	.63469										
0.7	1 04131	1.03037	.99794	91049	.87742	.80030	72143	.61442	.48061									
0.8	.96125	.95175	.92386	87928	.82136	.75571	.68809	.59398										
0.9	.89072	.88251	.85856	.82616	.77950	.71495	.65677	.57361										
1	.82843	.85005	.80465	.76809	.72587	.67769	62701	.55364	.45122	.31877	.24386	.19673	16516	.14182	.12394	09952	08292	07104
1.2	.72410	.71882	70370	67937	.64814	.61187	.57329	51552	43013	.31162	.24070	.19520	.16369	14099	12350			
1.5	.60555	60233	.57246	.57633	.55559	.53138	50496	46379	39872	.29945	.23495	.19053	.16199	14058	12281	09876	08270	07064
2	47214	.47022	.44512	.45656	.44502	.43202	41702	39242	35054	.27740	.22418	.18618	15846	13762	12124	09792	08196	07026
2 5	.38518	.38403	.38098	.37608	36940	36155	35243	.33698	30913	.25550	.21208	17898	15395	13463	11928	09700	08115	06980
3	32457	.32403	.32184	31887	.31464	30969	30381	.29364	27453	23487	.19977	17154	14919	13119	.11694	09558	.08061	06897
4	24620	24588	24820	.25128	24168	23932	23668	23164	22188	19908	.17640	15596	13864	12396	.11172	09300	.07864	06848
5	.19805	.19785				.19455			18450	17080	.15575	14130	12785	11615	10585	08915	07675	06695
6	16554					.16326			.15750	14868	13842	12792	11778	10836	.09990	08562	.07452	06522
7	14217					14077			.13699	.13097	.12404	11620	10843	.10101	09387	08197	07210	06377
8	12448					12352			12112	.11680	.11176	.10600	09976	.09400	08848	.07800	06928	06200
9	.11079					.10989			10854	.10548	.10161	.09702	.09234	08784	08298	07407	06678	05976
10								.09900	09820	.09510	.09290	08980	08300	08180	07710			

The computations, however, involve elliptic functions and were made through use of values tabulated by Legendre in 1826 (4). Direct determination of elliptic functions as part of a computer program is sufficiently complex and time consuming that extensive results of the type being presented herein have not been programmed, computed, and reported.

The simple relations which combine the tabulated functions and Poisson's ratio to give stresses, strains, or deflections are

Bulk stress

$$\theta = 2p(1+\nu)A = \sigma_z + \sigma_\rho + \sigma_\theta = e\frac{E_m}{1-2\nu} \tag{1}$$

Vertical stress

$$\sigma_z = p\left[A + B\right] \tag{2}$$

Radial horizontal stress

$$\sigma_\rho = p\left[2\nu A + C + (1-2\nu)F\right] \tag{3}$$

Tangential horizontal stress

$$\sigma_\theta = p\left[2\nu A - D + (1-2\nu)E\right] \tag{4}$$

Vertical-radial shear stress

$$\tau_{\rho z} = \tau_{z\rho} = pG \tag{5}$$

Bulk strain

$$e = p\frac{2(1+\nu)}{E_m}(1-2\nu)A = \epsilon_z + \epsilon_\rho + \epsilon_\theta = \theta\frac{1-2\nu}{E_m} \tag{6}$$

Vertical strain

$$\epsilon_z = p\frac{1+\nu}{E_m}\left[(1-2\nu)A + B\right] \tag{7}$$

Radial horizontal strain

$$\epsilon_\rho = p\frac{1+\nu}{E_m}\left[(1-2\nu)F + C\right] \tag{8}$$

Tangential horizontal strain

$$\epsilon_\theta = p\frac{1+\nu}{E_m}\left[(1-2\nu)E - D\right] \tag{9}$$

Vertical-radial shear strain

$$\gamma_{\rho z} = \gamma_{z\rho} = p\frac{2(1+\nu)}{E_m}G = \frac{2(1+\nu)}{E_m}\tau_{\rho z} \tag{10}$$

Vertical deflection

$$\omega_z = p\frac{1+\nu}{E_m}r\left[zA + (1-\nu)H\right] \tag{11}$$

Radial horizontal deflection

$$\omega_\rho = p\frac{1+\nu}{E_m}(-\rho r)\left[(1-2\nu)E - D\right] = -\rho r\epsilon_\theta \tag{12}$$

Tangential horizontal deflection

$$\omega_\theta = 0 \tag{13}$$

Symbols and sign conventions follow Timoshenko (6). Figure 1 may give a clearer concept of most of the symbols.

A, B, C ... H = functions, whose values are given in Tables 1 through 8;
z, ρ, θ = coordinate distances in radii, cylindrical coordinates—positive to the right, outward, and downward;
σ = normal stress—positive for compression;
σ_z, σ_ρ, σ_θ = letter subscript indicates axis parallel to which the line of action of the stress lies;
Θ = sum of three mutually perpendicular normal stresses at a point;
τ = shear stress—positive in positive quadrant and negative in mixed quadrant;
$\tau_{\rho z}$ = letter subscripts for shear stresses indicate (a) direction perpendicular to plane on which stress acts, and (b) the direction in which it acts;
γ = shearing strain;
$\gamma_{\rho z}$ = subscripts for shearing strain have same meanings as those for shear stress;
ϵ = strain—positive for compression;
ϵ_z, ϵ_ρ, ϵ_θ = subscripts for strain have same meanings as those for normal stress;
e = sum of three mutually perpendicular strains at a point;
ω = deflection—positive downward and outward;
ω_z, ω_ρ, ω_θ = subscripts for deflection have same meanings as those for normal stress;
ν = Poisson's ratio;
E_m = modulus of elasticity in compression—always positive;
p = surface contact pressure; and
r = radius of circular loaded area.

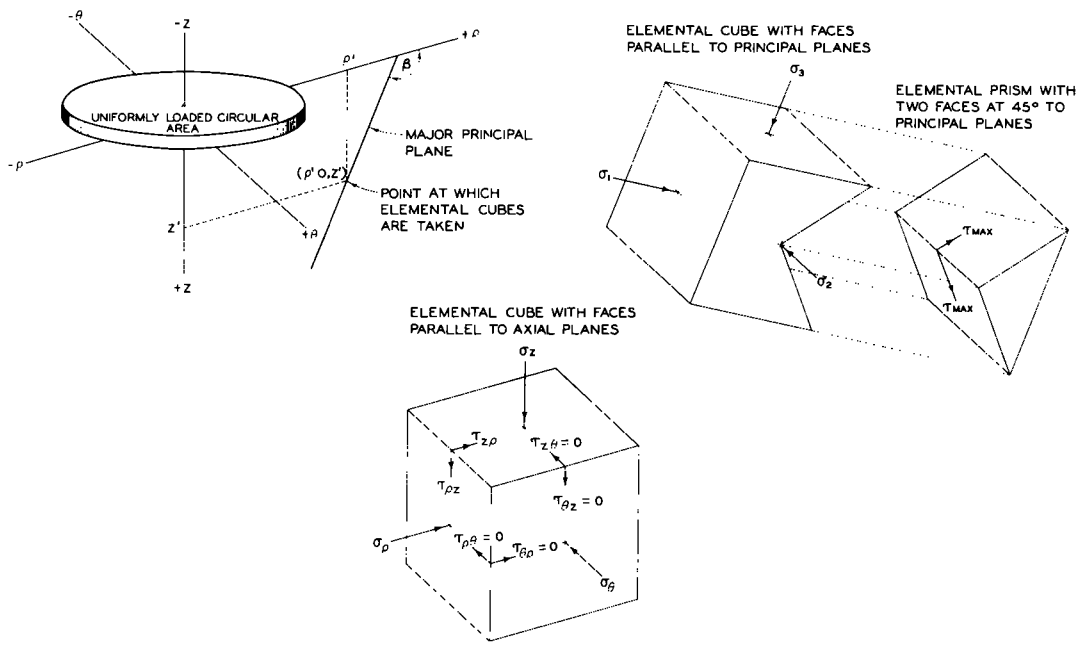

Figure 1. Stress directions.

From the symmetry of the circular load, it can readily be seen that there is no shear on vertical planes through the load axis. It follows that these must be principal planes; therefore, the tangential-horizontal stress, σ_ρ, must be a principal stress (the same is true for strain). The remaining two principal stresses may be obtained from the following expression:

$$\sigma_{1,\,2,\,3} = \frac{(\sigma_z + \sigma_\rho) \pm \sqrt{(\sigma_z - \sigma_\rho)^2 + (2\tau_{\rho z})^2}}{2}$$

in which $\sigma_{1,\,2,\,3}$ are principal stresses. Maximum shear stress, τ_{max}, may be obtained from the expression:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

These expressions can be found elsewhere (6, 7, or in almost any textbook on mechanics). They are included here for ready reference.

As an example of the use of the functions and equations presented, assume that vertical stress, strain, and deflection values are desired at a depth of 3.0 radii and offset of 6.0 radii for a Poisson's ratio of 0.3, modulus of elasticity of 10,000 psi, and surface contact pressure of 100 psi for a circular loaded area of 10-in. radius. Vertical stress, σ_z, will be determined from Eq. 2. Taking the value of function A from Table 1, for 3 radii depth and 6 radii offset, to be 0.00505 and for B from Table 2 to be -0.00192 and substituting it in Eq. 2,

$$\sigma_z = 100\,(0.00505 - 0.00192) = 0.313 \text{ psi}$$

Vertical strain, ϵ_z, is determined from Eq. 7. Inasmuch as functions A and B again apply, the preceding values again are used. Substituting them in Eq. 7,

$$\epsilon_z = 100\,\frac{1 + 0.3}{10,000}\,\Big[(1 - 0.6)0.00505 - 0.00192\Big] = 13 \times 10^{-7} \text{ in./in.}$$

Vertical deflection, ω_z, is determined from Eq. 11. Function A again applies, but function H must be obtained for 3 radii depth and 6 radii offset in Table 8. The value is 0.14919. Substituting in Equation 11,

$$\omega_z = 100\,\frac{1 + 0.3}{10,000}\;10\Big[(3)0.00505 + (1 - 0.3)\,0.14919\Big] = 0.015545 \text{ in.}$$

In summary, this paper presents tabular values for eight functions that can be combined in simple equations to give values of coordinate normal or shear stress or strain or of coordinate deflections for any desired value of Poisson's ratio for a uniformly distributed, circular load on an elastic, homogeneous, isotropic half space. Values are presented to five decimal place accuracy for each radius of offset from the load axis up to 14 and for each radius of depth up to 9. Values for fractional radii offset and depth are given to depths up to 4 radii for points beneath and just out from under the load. The paper is being presented for use as a reference to the computational results included rather than for the more usual purpose of reporting recent research findings.

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